Linear Algebra 2:
HW#1

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Summer 2025

due Friday, March 7, 2025, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and it should be either typed or written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Unless explicitly stated otherwise, make sure you **show your work** (though you may omit the details of row reduction, as per the Remark below). If you do not show your work, you will receive **no credit** for the exercise/problem in question, even if your final answer is correct.

Remark: Feel free to use a calculator such as the one from

`https://www.dcode.fr/matrix-row-echelon`

for any row reduction that you need to perform. However, keep in mind that on the exam, you might be asked to solve problems like this **without** a calculator, and so you should in principle be able to perform row reduction of this sort by hand.[a] Moreover, make sure you correctly tell the calculator what kinds of numbers you are working with,[b] because otherwise, the calculator may give you a wrong answer.

[a] If you do not yet feel confident in your ability to row reduce, then row reduce by hand first, and then check your answer with a calculator.

[b] For real numbers, you should use their "Echelon Form Matrix Reduction Calculator." For $\mathbb{Z}_2$, use "RREF with Modulo Calculator" with Base/Modulus = 2. For $\mathbb{Z}_3$, use "RREF with Modulo Calculator" with Base/Modulus = 3.

Remark: When working with coordinate vectors and matrices of linear functions, make sure you always **specify the bases** that you are working with.

Exercise 1 (10 points)**.** *Let U and V be real vector spaces (i.e. vector spaces over $\mathbb{R}$), let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ be a basis of U, and let $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be a basis of V. Let $f : U \to V$ be the unique linear function that satisfies the following:*

- $f(\mathbf{b}_1) = 7\mathbf{c}_1 - 2\mathbf{c}_3$;
- $f(\mathbf{b}_2) = \mathbf{c}_2$;
- $f(\mathbf{b}_3) = -\mathbf{c}_1 + 2\mathbf{c}_2$;
- $f(\mathbf{b}_4) = 3\mathbf{c}_2 - \mathbf{c}_3$.

(The existence and uniqueness of f follow from Theorem 4.3.2 of the Lecture Notes.) Compute the matrix ${}_{\mathcal{C}}\big[\ f\ \big]_{\mathcal{B}}$. Then, compute rank$(f)$ and dim$\big(Ker(f)\big)$. Is f one-to-one? Is it onto? Is it an isomorphism? (Make sure you justify your answer.)

Exercise 2 (10 points)**.** *Consider the following sets of vectors (with entries understood to be in $\mathbb{Z}_3$):*

- $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$;
- $\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Prove that $\mathcal{B}$ and $\mathcal{C}$ are bases of $\mathbb{Z}_3^3$, and compute the change of basis matrix ${}_{\mathcal{C}}\Big[\ Id_{\mathbb{Z}_3^3}\ \Big]_{\mathcal{B}}$.

Problem 1 (30 points)**.** *Let U and V be non-trivial, finite-dimensional vector spaces over a field $\mathbb{F}$, and let $f : U \to V$ be a linear function. Prove that the following are equivalent:*

(1) f is an isomorphism;

(2) there exists a positive integer n, a basis $\mathcal{B}$ of U, and a basis $\mathcal{C}$ of V such that ${}_{\mathcal{C}}\big[\ f\ \big]_{\mathcal{B}} = I_n$.

Problem 2 (30 points)**.** *Consider the following polynomials with coefficients in $\mathbb{Z}_2$ and matrices with entries in $\mathbb{Z}_2$:*

- $p_1(x) = x^3 + x$;
- $p_2(x) = x^2 + 1$;
- $p_3(x) = x^3+x^2+x+1$;
- $p_4(x) = x^2 + x$;
- $p_5(x) = x^3 + x + 1$;
- $p_6(x) = x^2$;

- $M_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$;
- $M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$;
- $M_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$;
- $M_4 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$;
- $M_5 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$;
- $M_6 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

(a) *Prove that there exists a unique linear function $f : \mathbb{P}^3_{\mathbb{Z}_2} \to \mathbb{Z}_2^{2\times 2}$ satisfying the property that $f\big(p_i(x)\big) = M_i$ for all indices $i \in \{1, \dots, 6\}$.*

> ***Notation:*** *As usual, $\mathbb{P}^3_{\mathbb{Z}_2}$ is the vector space (over $\mathbb{Z}_2$) of all polynomials with coefficients in $\mathbb{Z}_2$ and of degree at most 3. $\mathbb{Z}_2^{2\times 2}$ is the vector space (over $\mathbb{Z}_2$) of all 2×2 matrices with entries in $\mathbb{Z}_2$.*

(b) *Find a formula for the linear function f from part (a). Your final answer should be of the following form:*

$$f\big(a_3x^3 + a_2x^2 + a_1x + a_0\big) \quad = \quad \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \qquad \forall a_0, a_1, a_2, a_3 \in \mathbb{Z}_2$$

with the question marks replaced with the appropriate values.

(c) *Prove that the linear function f from part (a) is an isomorphism, and find a formula for f^{-1}. Your final answer should be of the following form:*

$$f^{-1}\Big(\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}\Big) \quad = \quad \underline{\hspace{3cm}} \qquad \forall a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2} \in \mathbb{Z}_2$$

with the blank filled in with the appropriate polynomial.

Problem 3 (20 points)**.** *Consider the following polynomials with coefficients in* $\mathbb{Z}_3$*:*

- $p_1(x) = x^2 + 2x$;
- $p_2(x) = x^3$;
- $p_3(x) = x + 1$;
- $p_4(x) = x^3 + 2x^2 + 2x + 1$;
- $p_5(x) = 2x^3 + x + 1$;
- $q_1(x) = 2$;
- $q_2(x) = x^3 + 1$;
- $q_3(x) = 2x^3 + x^2 + 2x + 1$;
- $q_4(x) = x^2 + 2x$;
- $q_5(x) = x^3 + x^2 + 2x$.

Prove that there exists a linear function $f : \mathbb{P}^3_{\mathbb{Z}_3} \to \mathbb{P}^3_{\mathbb{Z}_3}$ *satisfying the property that* $f\big(p_i(x)\big) = q_i(x)$ *for all indices* $i \in \{1, \ldots, 5\}$*. How many such linear functions* f *exist?*