Linear Algebra 2: Tutorial 11

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Exercise 1. *Consider the bilinear form f on $\mathbb{R}^3$ given by the formula*

$$f(\mathbf{x}, \mathbf{y}) \quad = \quad x_1y_1 + 2x_1y_2 + x_1y_3 - x_2y_1 + x_2y_2 + x_3y_1 - 5x_3y_3$$

for all $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T$ in $\mathbb{R}^3$. Compute the matrix of the bilinear form f with respect to the standard basis $\mathcal{E}_3$ of $\mathbb{R}^3$. Is the bilinear form f symmetric?

Exercise 2. *Compute the (symmetric) matrix of the quadratic form q on $\mathbb{R}^3$ given by*

$$q(\mathbf{x}) \quad = \quad x_1x_2 + 4x_1x_3 + x_2^2 - 6x_2x_3 + 2x_3^2$$

for all $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ in $\mathbb{R}^3$ (with respect to the standard basis $\mathcal{E}_3$ of $\mathbb{R}^3$).

Exercise 3. *Give an example of a quadratic form q on $\mathbb{Z}_2^2$ for which there does **not** exist a symmetric matrix $A \in \mathbb{Z}_2^{2\times 2}$ that satisfies the following:*

$$q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} \qquad \text{for all } \mathbf{x} \in \mathbb{Z}_2^2.$$

Exercise 4. *Determine whether the following matrices (with entries in $\mathbb{R}$ are positive definite. Do this in two ways: using the Gaussian elimination test of positive definiteness, and using Sylvester's criterion of positive definiteness.*

(a) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$;

(b) $B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$;

(c) $C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Exercise 5. *Consider the bilinear form* $\langle \cdot, \cdot \rangle$ *on* $\mathbb{R}^3$ *given by*

$$\langle \mathbf{x}, \mathbf{y} \rangle \quad = \quad 3x_1y_1 + 2x_1y_2 + x_1y_3 + 2x_2y_1 + 2x_2y_2 + x_3y_1 + 2x_3y_3$$

for all $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ *and* $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T$ *in* $\mathbb{R}^3$.

(a) Compute the matrix of the bilinear form $\langle \cdot, \cdot \rangle$ *with respect to the standard basis* $\mathcal{E}_4$ *of* $\mathbb{R}^4$.

(b) Is the bilinear form $\langle \cdot, \cdot \rangle$ *symmetric?*

(c) Is the bilinear form $\langle \cdot, \cdot \rangle$ *a scalar product in* $\mathbb{R}^3$*?*