Linear Algebra 2: Tutorial 10

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Summer 2024

Exercise 1. *Suppose you are given a matrix $A \in \mathbb{C}^{15\times 15}$, for which your friend the calculator computes the characteristic polynomial*

$$p_A(\lambda) \quad = \quad \lambda^7(\lambda-7)^2(\lambda+1)^6,$$

and further helpfully computes the following:

- $\mathit{rank}\big(A^0\big) = 15$;
- $\mathit{rank}\big(A^1\big) = 13$;
- $\mathit{rank}\big(A^2\big) = 11$;
- $\mathit{rank}\big(A^3\big) = 9$;
- $\mathit{rank}\big(A^4\big) = 8$;
- $\mathit{rank}\big(A^5\big) = 8$;

- $\mathit{rank}\big((A-7I_{15})^0\big) = 15$;
- $\mathit{rank}\big((A-7I_{15})^1\big) = 14$;
- $\mathit{rank}\big((A-7I_{15})^2\big) = 13$;
- $\mathit{rank}\big((A-7I_{15})^3\big) = 13$;

- $\mathit{rank}\big((A+I_{15})^0\big) = 15$;
- $\mathit{rank}\big((A+I_{15})^1\big) = 12$;
- $\mathit{rank}\big((A+I_{15})^2\big) = 9$;
- $\mathit{rank}\big((A+I_{15})^3\big) = 9$.

What is the Jordan normal form of A?

Exercise 2. *Suppose you are given a matrix $A \in \mathbb{C}^{7\times 7}$, for which your friend the calculator computes the characteristic polynomial*

$$p_A(\lambda) \quad = \quad (\lambda-2)^5(\lambda-5)^2,$$

and further helpfully computes the following:

- $\mathit{rank}\big((A-2I_7)^0\big) = 7$;
- $\mathit{rank}\big((A-2I_7)^1\big) = 5$;
- $\mathit{rank}\big((A-2I_7)^2\big) = 2$;
- $\mathit{rank}\big((A-2I_7)^3\big) = 2$;

- $\mathit{rank}\big((A-5I_7)^0\big) = 7$;
- $\mathit{rank}\big((A-5I_7)^1\big) = 5$;
- $\mathit{rank}\big((A-5I_7)^2\big) = 5$.

Oh, no!! The calculator messed up!!! (Or more likely: you miscopied something to and/or from the calculator.) How do you know that something went wrong?

Exercise 3. *Suppose that a matrix* $A \in \mathbb{C}^{10\times 10}$ *has (exactly) the following eigenvalues:*

- $\lambda_1 = 3$ *with algebraic multiplicity* 3 *and geometric multiplicity* 1*;*
- $\lambda_2 = 4$ *with algebraic multiplicity* 3 *and geometric multiplicity* 1*;*
- $\lambda_3 = -1$ *with algebraic multiplicity* 2 *and geometric multiplicity* 1*;*
- $\lambda_4 = 0$ *with algebraic multiplicity* 2 *and geometric multiplicity* 2*.*

Based on this information, can you determine the Jordan normal form of A *(up to a reordering of the Jordan blocks)? If not, determine all possible candidates that are consistent with the information above.*

Exercise 4. *Suppose that a matrix* $A \in \mathbb{C}^{12\times 12}$ *has (exactly) the following eigenvalues:*

- $\lambda_1 = -3$ *with algebraic multiplicity* 4 *and geometric multiplicity* 3*;*
- $\lambda_2 = -4$ *with algebraic multiplicity* 4 *and geometric multiplicity* 2*.*
- $\lambda_3 = -5$ *with algebraic multiplicity* 4 *and geometric multiplicity* 1*.*

Based on this information, can you determine the Jordan normal form of A *(up to a reordering of the Jordan blocks)? If not, determine all possible candidates that are consistent with the information above.*

Exercise 5. *For any subspace* V *of* $\mathbb{R}^n$*, let* P_V *be the matrix of orthogonal projection onto* V*. Is* P_V *orthogonally diagonalizable for every subspace* V *of* $\mathbb{R}^n$*? If so, explain why. If not, decide if it must be diagonalizable.*

Exercise 6. *Diagonalize the following matrix in* $\mathbb{R}^{n\times n}$ *(*$n \geq 2$*).*

$$A_n \quad = \quad \begin{bmatrix} 3 & 1 & \dots & 1 \\ 1 & 3 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 3 \end{bmatrix}.$$

> ***Hint:*** *Guess the eigenvalues, and then find bases of the corresponding eigenspaces. As a starting point, here are the characteristic polynomials of* A_n *for small values of* n*:*
>
> - $p_{A_2}(\lambda) = (\lambda - 4)(\lambda - 2)$
> - $p_{A_3}(\lambda) = (\lambda - 5)(\lambda - 2)^2$

- $p_{A_4}(\lambda) = (\lambda - 6)(\lambda - 2)^3$

Exercise 7. *Let p be any real number. Diagonalize the following matrix in $\mathbb{R}^{n\times n}$ ($n \geq 2$).*

$$B_n = \begin{bmatrix} p & 1 & \dots & 1 \\ 1 & p & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & p \end{bmatrix}.$$

Exercise 8. *Orthogonally diagonalize the following matrix in $\mathbb{R}^{n\times n}$ ($n \geq 2$).*

$$C_n = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

> ***Hint:*** *To make your Q, you might want to start with vectors $\mathbf{w}_k := \begin{bmatrix} 1 & \dots & 1 & -k & 0 & \dots & 0 \end{bmatrix}^T$ (k many 1's), plus another vector (which one?). Now what?*