Linear Algebra 2: Tutorial 9

Todor Antić & Irena Penev

Summer 2024

Exercise 1. *Let $\mathbb{F}$ be a field, and let $\lambda_0 \in \mathbb{F}$ be an eigenvalue of a square matrix $A \in \mathbb{F}^{n\times n}$.*

(a) Prove that for all non-negative integers m, λ_0^m is an eigenvalue of A^m.

(b) Prove that if A is invertible, then λ_0^m is an eigenvalue of A^m for all integers m.

- *This, in particular, means that $\frac{1}{\lambda_0}$ is an eigenvalue of A^{-1} (again, assuming that A is invertible). How do you know that $\frac{1}{\lambda_0}$ is even defined, i.e. that you are not dividing by zero?*

(c) Assume that $B := P^{-1}AP$ for some invertible matrix $P \in \mathbb{F}^{n\times n}$. In particular, A and B are similar, and so by Theorem 8.2.9 of the Lecture Notes, A and B have the same eigenvalues. If $\mathbf{v}$ is an eigenvector of A associated with the eigenvalue λ_0, can you construct an eigenvector of B associated with the eigenvalue λ_0?

Exercise 2. *Construct a matrix $A \in \mathbb{R}^{2\times 2}$ such that A has no (real) eigenvalues, but A^2 does have (real) eigenvalues.*

> ***Hint:*** *Think geometrically.*

Exercise 3. *For the following matrices D and P (with entries understood to be in $\mathbb{C}$), set $A := PDP^{-1}$ (so, $D = P^{-1}AP$), and then compute the characteristic polynomial and the spectrum of A, specify all the eigenvalues of A along with their algebraic and geometric multiplicities, and find a basis for each eigenspace of A. (In both parts, you may assume that the matrix P is indeed invertible. This can be checked, for example, by computing the determinant of P and seeing that it is non-zero.)*

(a) $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \end{bmatrix}$;

(b) $$D := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & 0 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 2 & 3 & -2 & -3 \\ 4 & 5 & 6 & 7 \end{bmatrix}.$$

Exercise 4. *For each of the following matrices A, determine whether the matrix is diagonalizable, and if so, diagonalize it. If A is diagonalizable, then find a formula for A^m for all non-negative integers m.*[1] *Finally, determine whether your formula for A^m also works for negative integers m.*

(a) $$A = \begin{bmatrix} -2 & -10 & 0 \\ 0 & 3 & 0 \\ -5 & -10 & 3 \end{bmatrix}$$

(b) $$A = \begin{bmatrix} -2 & 4 & 1 \\ -2 & 4 & 1 \\ -2 & 2 & 1 \end{bmatrix}$$

(c) $$A = \begin{bmatrix} 6 & 3 & 1 \\ -8 & -4 & -2 \\ 8 & 6 & 4 \end{bmatrix}$$

(d) $$A = \begin{bmatrix} 4 & -3 & 3 & 0 \\ 0 & 11 & 0 & -6 \\ -2 & 12 & -1 & -6 \\ 0 & 18 & 0 & -10 \end{bmatrix}$$

[1] If A is not diagonalizable, then you do not need to find a formula for A^m.