Linear Algebra 2: Tutorial 8

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Summer 2024

Exercise 1. *For each of the following matrices A in $\mathbb{C}^{3\times 3}$, compute the characteristic polynomial $p_A(\lambda)$ and the spectrum of A. For each eigenvalue λ of A, determine both the algebraic and the geometric multiplicity of λ, and compute a basis of the eigenspace E_λ.*

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$;

(b) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$;

(c) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$;

(d) $A = \begin{bmatrix} 3 & 0 & -2 \\ -7 & 0 & 4 \\ 4 & 0 & -3 \end{bmatrix}$;

(e) $A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 0 & 1 \\ -4 & 0 & 3 \end{bmatrix}$;

Definition. *For a field $\mathbb{F}$, a scalar $\lambda_0 \in \mathbb{F}$, and a positive integer t, the* Jordan block $J_t(\lambda_0)$ *is defined to be following $t \times t$ matrix (with entries understood to be in $\mathbb{F}$):*

$$J_t(\lambda_0) \quad = \quad \begin{bmatrix} \lambda_0 & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_0 & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_0 \end{bmatrix}_{t \times t}.$$

Thus, $J_t(\lambda_0)$ is a matrix in $\mathbb{F}^{t\times t}$, it has all λ_0's on the main diagonal, all 1's on the diagonal right above the main diagonal, and 0's everywhere else.

Exercise 2. *Let $\mathbb{F}$ be a field, let $\lambda_0 \in \mathbb{F}$, and let t be a positive integer. What are the eigenvalues of $J_t(\lambda_0)$? What are their algebraic and geometric multiplicities? Find a basis of each eigenspace.*

Definition. *Let $\mathbb{F}$ be a field. For square matrices $A_1 \in \mathbb{F}^{n_1\times n_1}, A_2 \in \mathbb{F}^{n_2\times n_2}, \ldots, A_k \in \mathbb{F}^{n_k\times n_k}$, we define the* direct sum *of $A_1, A_2 \ldots, A_k$ to be the $(n_1+n_2+\cdots+n_k)\times(n_1+n_2+\cdots+n_k)$ matrix*

$$A_1 \oplus A_2 \oplus \cdots \oplus A_k \quad := \quad \left[\begin{array}{c:c:c:c} A_1 & O_{n_1\times n_2} & \cdots & O_{n_1\times n_k} \\ \hdashline O_{n_2\times n_1} & A_2 & \cdots & O_{n_2\times n_k} \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline O_{n_k\times n_1} & O_{n_k\times n_2} & \cdots & A_k \end{array}\right].$$

For example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \oplus \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix} \quad = \quad \left[\begin{array}{cc:ccc:c} 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 5 & 6 & 0 \\ 0 & 0 & 7 & 8 & 9 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Exercise 3. *Let $\mathbb{F}$ be a field, let $\lambda_0 \in \mathbb{F}$, and let $t_1, \ldots, t_k$ be positive integers. What are the eigenvalues of $J_{t_1}(\lambda_0) \oplus \cdots \oplus J_{t_k}(\lambda_0)$? What are the algebraic and geometric multiplicities of those eigenvalues?*

Exercise 4. *Let $\mathbb{F}$ be a field, let $\lambda_1, \lambda_2 \in \mathbb{F}$ be distinct, and let t_1, t_2 be positive integers. What are the eigenvalues of $J_{t_1}(\lambda_1) \oplus J_{t_2}(\lambda_2)$? What are the algebraic and geometric multiplicities of those eigenvalues?*

Exercise 5. *Try to generalize Exercises 3 and 4. What are the eigenvalues of matrix that is a direct sum of arbitrarily many Jordan blocks, when those Jordan blocks can be of arbitrary size and type?[1] (Such a matrix is called a "Jordan matrix," or a matrix in "Jordan normal form.") What are the algebraic and geometric multiplicities of those eigenvalues?*

> ***Remark:*** *You don't have to give a fully formal proof, but try to give a reasonable proof outline.*

[1] So, the matrix is of the form $J_{t_1}(\lambda_1) \oplus \cdots \oplus J_{t_k}(\lambda_k)$, where $\lambda_1, \ldots, \lambda_k$ are (not necessarily distinct) scalars in the field $\mathbb{F}$ in question, and $t_1, \ldots, t_k$ are some positive integers.