Linear Algebra 2: Tutorial 7

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Theorem 7.11.1. *Let $\mathbb{F}$ be an **algebraically closed field**. Let m and n be positive integers, and let $p(x) = \sum_{i=0}^{m} a_i x^i$ ($a_m \neq 0$) and $q(x) = \sum_{i=0}^{n} b_i x^i$ ($b_n \neq 0$) be polynomials with coefficients in $\mathbb{F}$. Let P be the $n \times (n+m)$ matrix whose j-th row (for $j \in \{1, \dots, n\}$) is*

$$\Big[\ \underbrace{0\ \dots\ 0}_{j-1}\ \ a_m\ \ a_{m-1}\ \ \dots\ \ a_0\ \ \underbrace{0\ \dots\ 0}_{n-j}\ \Big],$$

and let Q be the $m \times (n+m)$ matrix whose j-th row (for $j \in \{1, \dots, m\}$) is

$$\Big[\ \underbrace{0\ \dots\ 0}_{j-1}\ \ b_n\ \ b_{n-1}\ \ \dots\ \ b_0\ \ \underbrace{0\ \dots\ 0}_{m-j}\ \Big].$$

Then $p(x)$ and $q(x)$ have a common root in $\mathbb{F}$ if and only if

$$det\left(\begin{bmatrix} P \\ \hline Q \end{bmatrix} \right) = 0.$$

Exercise 1. *In this exercise we use the notation from Theorem 7.11.1.*

(a) Analyze the proof of Theorem 7.11.1, and determine whether both implications require an algebraically closed field, or if only one (which one?) does.

*(b) Suppose that $\mathbb{F}$ is **any** field (not necessarily an algebraically closed one). Is either one of the statements below guarantees to be true? (Can you come up with a counterexample to one (or both) of the implications?)*

1. *If $det\left(\begin{bmatrix} P \\ \hline Q \end{bmatrix} \right) = 0$, then $p(x)$ and $q(x)$ have a common root in $\mathbb{F}$.*
2. *If $det\left(\begin{bmatrix} P \\ \hline Q \end{bmatrix} \right) \neq 0$, then $p(x)$ and $q(x)$ do **not** have a common root in $\mathbb{F}$.*

Exercise 2. *For invertible matrices $A, B \in \mathbb{R}^{n\times n}$, what is the relationship between adj(A), adj(B), and adj(AB)?*

Exercise 3. *Prove or disprove the following statement:*

> *For all matrices $A = \begin{bmatrix} \mathbf{a}_1 & \dots \mathbf{a}_n \end{bmatrix}$ in $\mathbb{R}^{n\times n}$, we have that $|det(A)| \leq \prod_{i=1}^{n} ||\mathbf{a}_i||$.*
>
> ***Hint:*** *Volume.*

Exercise 4. *What is the maximum possible value of det(A) if A is a matrix in $\mathbb{R}^{4\times 4}$, all of whose entries are 1, 0, or −1? Exhibit a matrix A for which this maximum is reached.*

Exercise 5. *Prove of disprove the following statement:*

> *For all invertible matrices $A \in \mathbb{R}^{2\times 2}$ and $\mathbf{v} \in \mathbb{R}^2$, we have that $||A\mathbf{v}|| \leq |det(A)|\ ||\mathbf{v}||$.*

Exercise 6. *Either construct a matrix $A \in \mathbb{R}^{3\times 3}$ such that $A^2 = -I_3$, or prove that no such matrix exists.*

Exercise 7.

(a) *Either construct invertible matrices $P, A \in \mathbb{R}^{3\times 3}$ such that $P^{-1}AP = -A$, or prove that no such matrices exist.*

(b) *Either construct invertible matrices $P, A \in \mathbb{R}^{3\times 3}$ such that $P^T AP = -A$, or prove that no such matrices exist.*

Exercise 8. *Prove or disprove the each of the following statements.*

(a) *For all $A, B \in \mathbb{R}^{2\times 2}$, $det(A + B) \neq det(A) + det(B)$.*

(b) *For all $A \in \mathbb{R}^2$, there exists some $B \in \mathbb{R}^{2\times 2}$ such that $det(A + B) \neq det(A) + det(B)$.*

Exercise 9. *Show that the area of the triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) in $\mathbb{R}^2$ is equal to*

$$\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|.$$

Exercise 10. *For which (if any) real values of k is the matrix*

$$A = \begin{bmatrix} k^2 & 1 & 4 \\ k & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

invertible?