Linear Algebra 2: Tutorial 6

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Exercise 1. *Using the* ***definition*** *of a determinant (with permutations), compute the determinant of the matrix below.*

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

Exercise 2. *Consider the matrix A below, with entries understood to be in $\mathbb{R}$. Determine whether det(A) is positive, negative, or zero. Can you tell which it is without actually computing the determinant?*

$$A = \begin{bmatrix} 1 & 1000 & 2 & 3 \\ 1000 & -3 & 5 & 0 \\ 2 & 3 & 5 & 1000 \\ 1 & 2 & 1000 & 4 \end{bmatrix}$$

Exercise 3. *Compute the determinant of the matrix below, with entries understood to be in $\mathbb{Z}_3$.*

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

Exercise 4. *Let $a, b, c, d, e, f, g, h, i$ be real numbers such that*

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$$

Compute the following determinants.

1. $\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix}$

2. $\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix}$

3. $\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$

4. $\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}$

5. $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$

6. $\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix}$

Exercise 5. *Compute the determinants of the matrices below, with entries understood to be in* $\mathbb{R}$*.*

- $A_1 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$
- $A_2 = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 4 & 6 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$
- $A_3 = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 2 & 0 & -2 & 4 \\ 3 & 3 & -3 & 3 \\ 4 & 4 & -4 & 4 \end{bmatrix}$
- $A_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$
- $A_5 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$
- $A_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}$

Exercise 6. *Let A be an $n \times n$ matrix of the following form:*

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_{1,n} \\ 0 & 0 & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & a_{n,n} \end{bmatrix}$$

Find a formula for det(A).

Exercise 7. *Using determinants, determine for which (if any) values of k the matrix A (below) is invertible. (Entries of A are understood to be in $\mathbb{R}$).*

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & k & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

Exercise 8. *Let $A \in \mathbb{R}^{n\times n}$. If $det(A) = 3$, then what is $det(A^T A)$?*

Exercise 9. *Let $A \in \mathbb{R}^{n\times n}$. What can you say about the sign of $det(A^T A)$?*

Definition. *Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$.*

- *A is* symmetric *if $A^T = A$.*
- *A is* skew-symmetric *of $A^T = -A$.*

Exercise 10. *Let $\mathbb{F}$ be a field of characteristic other than 2, i.e. assume that in the field $\mathbb{F}$, $1 + 1 \neq 0$.*

(a) Show that if n is odd positive integer, then any skew-symmetric matrix $A \in \mathbb{F}^{n\times n}$ is non-invertible. (Use determinants.)

(b) For each even positive integer n, construct an invertible skew-symmetric matrix in $\mathbb{F}^{n\times n}$.