Linear Algebra 2: Tutorial 5

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Theorem 6.6.3. *Let $A \in \mathbb{R}^{n\times m}$ be a matrix of rank m (i.e. A is a matrix of full column rank). Then the matrix $A(A^TA)^{-1}A^T$ is the standard matrix of orthogonal projection onto $Col(A)$, that is, for all $\mathbf{x} \in \mathbb{R}^n$, the orthogonal projection of $\mathbf{x}$ onto $C := Col(A)$ is given by*

$$\mathbf{x}_C \quad = \quad A(A^TA)^{-1}A^T\mathbf{x}.$$

Theorem 6.7.1. *Let $A \in \mathbb{R}^{n\times m}$ and $\mathbf{b} \in \mathbb{R}^n$. Then the matrix-vector equation*

$$A^TA\mathbf{x} \quad = \quad A^T\mathbf{b}$$

is consistent, and moreover, its solution set is precisely the set of vectors $\mathbf{x}$ in $\mathbb{R}^m$ that minimize the expression

$$||A\mathbf{x} - \mathbf{b}||.$$

The Interpolation Theorem. *For all pairwise distinct $x_0, x_1, \dots, x_n \in \mathbb{R}$ and all (not necessarily distinct) $y_0, y_1, \dots, y_n \in \mathbb{R}$, there exists a unique polynomial $p(x)$ of degree at most n such that $p(x_i) = y_i$ for all $i = 0, 1, \dots, n$.*

> ***Remark:*** *This theorem essentially states that if we are given $n+1$ data points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ (with $x_0, x_1, \dots, x_n$ pairwise distinct), then there is a unique polynomial of degree at most n whose graph passes through those data points.*

Proof. Omitted. □

Exercise 1. *Compute the matrices of the following permutations in S_4:*

(a) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$; *(b)* $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$.

Exercise 2. *Determine which of the following matrices are permutation matrices, and for those that are, find the permutations that correspond to them.*

(a) $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

(b) $B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$;

(c) $C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Exercise 3. *In what follows, we assume that* $\mathbb{R}^3$ *is equipped with the standard scalar product* $\cdot$. *Consider the following vectors in* $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Set $C := \mathit{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

(a) Compute the standard matrix P *of the orthogonal projection* $\mathit{proj}_C : \mathbb{R}^3 \to \mathbb{R}^3$ *onto* C.

> ***Warning:*** *The matrix* $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ *does* ***not*** *have rank* 3. *So, you cannot use Theorem 6.6.3 directly.*

(b) Using the matrix P *from part (a), compute the vector* $\mathbf{x}_C$ *(the projection of* $\mathbf{x}$ *onto* C*). Does* $\mathbf{x}$ *belong to* C*?*

Exercise 4. *Find a polynomial of degree at most* 3 *that passes through the points* $(1,3)$, $(-1,13)$, $(2,1)$, $(-2,33)$.

Exercise 5. *Fit a linear function of the the form* $f(t) = c_0 + c_1 t$ *to the data points* $(0,0)$, $(0,1)$, $(1,1)$ *using least squares.*

Exercise 6. *The following table gives world population in 10-year intervals, starting with 1950 and ending with 2000.*

year	*population (in 10^9)*
1950	*2.519*
1960	*2.982*
1970	*3.692*
1980	*4.435*
1990	*5.263*
2000	*6.070*

We would like to use the least-squares method to find the linear function that best fits these data. Set up the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ *to which we need to apply the least-squares method.*

Exercise 7. *Using the least-squares method, fit a quadratic function to the four data points* $(a_1, b_1) = (-1, 8)$, $(a_2, b_2) = (0, 8)$, $(a_3, b_3) = (1, 4)$, *and* $(a_4, b_4) = (2, 16)$.

Exercise 8. *Explain how one would a trigonometric function of the form* $f(t) = c_0 + c_1 \sin t + c_2 \cos t$ *that best fits the data points* $(0, 0)$, $(1, 1)$, $(2, 2)$, $(3, 3)$.

> ***Remark:*** *Here, you should set up the matrix-vector equation* $A\mathbf{c} = \mathbf{b}$ *(where* A *and* $\mathbf{b}$ *are known and* $\mathbf{c}$ *is the unknown) to which the least-squares method should be applied. Don't compute the actual least-squares solution.*