Linear Algebra 2: Tutorial 4

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Exercise 1. *Consider the following vectors in* $\mathbb{R}^3$*:*

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{a}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Set $U := Span(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$*. Compute an orthonormal basis of* U *and an orthonormal basis of* $U^\perp$*.*

Exercise 2. *Consider the scalar product* $\langle \cdot, \cdot \rangle$ *on* $\mathbb{R}^2$ *defined by*

$$\langle \mathbf{x}, \mathbf{y} \rangle \quad = \quad x_1 y_1 + 2 x_2 y_2$$

for all $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *and* $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ *in* $\mathbb{R}^2$*. (You may assume that* $\langle \cdot, \cdot \rangle$ *really is a scalar product in* $\mathbb{R}^2$*.) Set* $U := Span(\begin{bmatrix} 1 \\ 1 \end{bmatrix})$*. Find an orthonormal basis of* U *and an orthonormal basis of* $U^\perp$ *with respect to* $\langle \cdot, \cdot \rangle$ *and the induced norm* $|| \cdot ||$*.*

Exercise 3. *Let* $|| \cdot ||$ *be the norm induced by the standard scalar product* $\cdot$ *on* $\mathbb{R}^n$*. Define* $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *by setting*

$$\langle \mathbf{x}, \mathbf{y} \rangle \quad = \quad ||\mathbf{x} + \mathbf{y}||^2 - ||\mathbf{x}||^2 - ||\mathbf{y}||^2 \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Is $\langle \cdot, \cdot \rangle$ *a scalar product in* $\mathbb{R}^n$*? Justify your answer.*

Exercise 4.

(a) Let V *be a real vector space, equipped with the scalar product* $\langle \cdot, \cdot \rangle$*, and let* $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ *be an orthonormal basis of* V*. Let* $\cdot$ *be the standard scalar product in* $\mathbb{R}^n$*. Prove that for all* $\mathbf{x}, \mathbf{y} \in V$*, we have that*

$$\langle \mathbf{x}, \mathbf{y} \rangle \quad = \quad \left[\ \mathbf{x}\ \right]_{\mathcal{B}} \cdot \left[\ \mathbf{y}\ \right]_{\mathcal{B}}.$$

__Hint:__ The first thing you should do is find a formula for $[\ \mathbf{x}\ ]_{\mathcal{B}}$ *and* $[\ \mathbf{y}\ ]_{\mathcal{B}}$ *using the orthonormal basis* $\mathcal{B}$ *and the scalar product* $\langle \cdot, \cdot \rangle$ *in* V.

(b) Same question as in part (a), only for a complex vector space V.

Exercise 5. *Let* $\langle \cdot, \cdot \rangle$ *be a scalar product in* $\mathbb{R}^n$ *(not necessarily the standard one). Let* V *be a real vector space, and let* $f : V \to \mathbb{R}^n$ *be a linear transformation. We define* $\langle \cdot, \cdot \rangle_f : V \times V \to \mathbb{R}$ *by setting*

$$\langle \mathbf{u}, \mathbf{v} \rangle_f \quad = \quad \Big\langle f(\mathbf{u}), f(\mathbf{v}) \Big\rangle$$

for all $\mathbf{u}, \mathbf{v} \in V$. *Prove that* $\langle \cdot, \cdot \rangle_f$ *is a scalar product in* V *if and only if* f *is one-to-one.*