Linear Algebra 2: Tutorial 3

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Definition. *Suppose we are given a real or complex vector space V, equipped with a scalar product $\langle \cdot, \cdot \rangle$. For a* ***non-zero*** *vector $\mathbf{u} \in V$ and any vector $\mathbf{v} \in V$, the* orthogonal projection *of $\mathbf{v}$ onto $\mathbf{u}$ is the vector*

$$proj_{\mathbf{u}}(\mathbf{v}) \quad := \quad \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u}.$$

Exercise 1. *Suppose that V is a real of complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$, and suppose that $\mathbf{u} \in V \setminus \{\mathbf{0}\}$.*

(a) If $\mathbf{v} \in V$ is a scalar multiple of $\mathbf{u}$, what is $proj_{\mathbf{u}}(\mathbf{v})$?

(b) If $\mathbf{w} \in V$ is orthogonal to $\mathbf{u}$, what is $proj_{\mathbf{u}}(\mathbf{w})$?

In both parts, justify your answer formally (using the definition of orthogonal projection), and draw a picture to give the intuition behind your answer.

Exercise 2. *Consider the following linearly independent vectors in $\mathbb{R}^4$:*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}.$$

Compute an orthonormal basis of $U := Span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ in two different ways: using Gram-Schmidt orthogonalization (version 1) and using Gram-Schmidt orthogonalization (version 2).

Exercise 3. *Suppose that V is a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$, and suppose that $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent vectors in V. Now suppose that you perform the two versions of Gram-Schmidt orthogonalization on the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in order to obtain orthonormal bases of $U := Span(\mathbf{v}_1, \dots, \mathbf{v}_k)$. Under what circumstances will your two orthonormal bases be different? (Will they ever be different?) Justify your answer.*

Exercise 4. *Consider the vectors* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *from Exercise 2. How would you obtain an orthonormal basis for* $U := \mathit{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ *that is* ***different*** *from the one(s) that you obtained in Exercise 2?*

> ***Remark:*** *Your basis should differ from the one(s) from Exercise 2 not merely in the order in which the vectors appear in the basis, but also in terms of the actual vectors that the basis contains. Don't perform the whole calculation: just explain how you would compute.*

Exercise 5. *Suppose that* V *is a real or complex vector space, equipped with a scalar product* $\langle \cdot, \cdot \rangle$ *and the induced norm* $||\cdot||$*. Suppose that* $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *is an orthogonal set of non-zero vectors in* V*. What do you obtain if you perform the Gram-Schmidt orthogonalization process (version 1) on* $\mathbf{v}_1, \ldots, \mathbf{v}_k$ ***without*** *normalizing at the end? And what happens if you do normalize at the end? What happens if you perform the Gram-Schmidt orthogonalization process (version 2) on* $\mathbf{v}_1, \ldots, \mathbf{v}_k$*?*

Exercise 6. *Consider the following vectors in* $\mathbb{R}^7$*:*

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 9 \\ 27 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -2 \\ 4 \\ -6 \\ 8 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Compute an orthonormal basis of $U := \mathit{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$*. Can you tell what the answer will be without actually performing Gram-Schmidt orthogonalization (either version)?*