Linear Algebra 2: Tutorial 2

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Exercise 1. *Compute the angle θ between the following two vectors in $\mathbb{R}^4$:*

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}.$$

What kind of angle (acute, right, or obtuse) do you get?

Exercise 2. *Compute the angle θ between the following two vectors in $\mathbb{R}^n$:*

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

What values do you get for θ when $n = 2, 3, 4$? (When do you get a nice angle and when do you need to settle for an expression involving an inverse trigonometric function?) Find the limit of θ as n approaches infinity.

Exercise 3. *Find the set of all vectors that are (simultaneously) orthogonal to the following three vectors in $\mathbb{R}^4$ (where orthogonality is assumed to be with respect to the standard scalar product $\cdot$ in $\mathbb{R}^4$):*

$$\mathbf{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}.$$

Exercise 4. *Generalize your answer to Exercise 3. More precisely, suppose that you are given vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$ in $\mathbb{R}^n$. How would you compute the set of all vectors in $\mathbb{R}^n$ that are simultaneously orthogonal (with respect to the standard scalar product $\cdot$) to the vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$? Must the set that you obtain be a subspace of $\mathbb{R}^n$?*

Exercise 5. *By carefully examining the proof of the Cauchy-Schwarz inequality (copied and pasted from the Lecture Notes below), determine when that inequality becomes an equality.*

The Cauchy–Schwarz inequality. *Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$. Then*

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \quad \leq \quad ||\mathbf{x}||\,||\mathbf{y}||$$

for all $\mathbf{x}, \mathbf{y} \in V$.

Proof. Fix $\mathbf{x}, \mathbf{y} \in V$. We may assume that $\langle \mathbf{x}, \mathbf{y} \rangle \neq 0$, for otherwise, the result is immediate. Note that this implies that $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$, and consequently, $||\mathbf{x}||, ||\mathbf{y}|| \neq 0$. We set

$$\mathbf{z} \quad := \quad \tfrac{\langle \mathbf{y},\mathbf{y} \rangle}{\langle \mathbf{x},\mathbf{y} \rangle}\mathbf{x} - \mathbf{y},$$

and we compute

$$\langle \mathbf{z}, \mathbf{y} \rangle \quad = \quad \langle \tfrac{\langle \mathbf{y},\mathbf{y} \rangle}{\langle \mathbf{x},\mathbf{y} \rangle}\mathbf{x} - \mathbf{y}, \mathbf{y} \rangle \quad \overset{(*)}{=} \quad \tfrac{\langle \mathbf{y},\mathbf{y} \rangle}{\langle \mathbf{x},\mathbf{y} \rangle}\langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{y}, \mathbf{y} \rangle \quad = \quad 0,$$

where (*) follows from r.2 and r.3 if V is a real vector space, or from c.2 and c.3 if V is a complex vector space. We have now shown that $\mathbf{z} \perp \mathbf{y}$, and so by the Pythagorean theorem, we have that

$$||\mathbf{z} + \mathbf{y}||^2 \quad = \quad ||\mathbf{z}||^2 + ||\mathbf{y}||^2.$$

But by construction, $\mathbf{z} + \mathbf{y} = \frac{\langle \mathbf{y},\mathbf{y} \rangle}{\langle \mathbf{x},\mathbf{y} \rangle}\mathbf{x}$, and consequently:

$$||\mathbf{z} + \mathbf{y}|| \quad = \quad ||\tfrac{\langle \mathbf{y},\mathbf{y} \rangle}{\langle \mathbf{x},\mathbf{y} \rangle}\mathbf{x}|| \quad \overset{(*)}{=} \quad |\tfrac{\langle \mathbf{y},\mathbf{y} \rangle}{\langle \mathbf{x},\mathbf{y} \rangle}|\,||\mathbf{x}|| \quad = \quad \tfrac{|\langle \mathbf{y},\mathbf{y} \rangle|}{|\langle \mathbf{x},\mathbf{y} \rangle|}\,||\mathbf{x}|| \quad = \quad \tfrac{||\mathbf{y}||^2}{|\langle \mathbf{x},\mathbf{y} \rangle|}\,||\mathbf{x}||,$$

where (*) follows from Proposition 6.2.1. So,

$$\tfrac{||\mathbf{y}||^4}{|\langle \mathbf{x},\mathbf{y} \rangle|^2}\,||\mathbf{x}||^2 \quad = \quad ||\mathbf{z} + \mathbf{y}||^2 \quad = \quad ||\mathbf{z}||^2 + ||\mathbf{y}||^2 \quad \geq \quad ||\mathbf{y}||^2,$$

which yields

$$\tfrac{||\mathbf{y}||^4}{|\langle \mathbf{x},\mathbf{y} \rangle|^2}\,||\mathbf{x}||^2 \quad \geq \quad ||\mathbf{y}||^2.$$

Since $\langle \mathbf{x}, \mathbf{y} \rangle$ and $||\mathbf{y}||$ are both non-zero, we have that $\frac{|\langle \mathbf{x},\mathbf{y} \rangle|^2}{||\mathbf{y}||^2}$ is defined and positive. So, we may multiply both sides of the inequality above by $\frac{|\langle \mathbf{x},\mathbf{y} \rangle|^2}{||\mathbf{y}||^2}$ to obtain

$$||\mathbf{x}||^2||\mathbf{y}||^2 \quad \geq \quad |\langle \mathbf{x}, \mathbf{y} \rangle|^2.$$

By taking the square root of both sides, we get

$$||\mathbf{x}||\,||\mathbf{y}|| \quad \geq \quad |\langle \mathbf{x}, \mathbf{y} \rangle|,$$

which is what we needed to show. □

Definition. *A* scalar product *(also called* inner product*) in a real vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies the following four axioms:*

r.1. for all $\mathbf{x} \in V$, $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, and equality holds if and only if $\mathbf{x} = \mathbf{0}$;

r.2. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$;

r.3. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{R}$, $\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha\langle \mathbf{x}, \mathbf{y} \rangle$;

r.4. for all $\mathbf{x}, \mathbf{y} \in V$, $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.

Exercise 6. *Which (if any) of the following functions $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ are scalar products in $\mathbb{R}^2$?*

(a) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 - x_2y_2$ *for all* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *and* $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ *in* $\mathbb{R}^2$;

(b) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + 2x_2y_2$ *for all* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *and* $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ *in* $\mathbb{R}^2$;

(c) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1x_2 + y_1y_2$ *for all* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *and* $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ *in* $\mathbb{R}^2$.

Make sure you justify your answer in each case. You should refer to the definition of the scalar product in a real vector space (above).