Linear Algebra 2: Tutorial 1

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Exercise 1. *Let U be a vector space over $\mathbb{Z}_2$, let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis of U, and consider the set $\mathcal{C} = \{\mathbf{b}_1 + \mathbf{b}_2, \mathbf{b}_2 + \mathbf{b}_3, \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3\}$.*

(a) Using coordinate vectors, prove that $\mathcal{C}$ is a basis of U.

(b) Compute the change of basis matrices ${}_{\mathcal{C}}\left[\ Id_U\ \right]_{\mathcal{B}}$ and ${}_{\mathcal{B}}\left[\ Id_U\ \right]_{\mathcal{C}}$.

(c) Compute the coordinate vectors $\left[\ \mathbf{b}_1 + \mathbf{b}_3\ \right]_{\mathcal{B}}$ and $\left[\ \mathbf{b}_1 + \mathbf{b}_3\ \right]_{\mathcal{C}}$.

> ***Remark:*** *You should be able to just "read off" one of these two vectors (which one?). For the other one, use a suitable change of basis matrix that you computed in part (b).*

(d) Explain why there exists a unique linear function $f : U \to U$ that satisfies all the following:

- $f(\mathbf{b}_1 + \mathbf{b}_2) = \mathbf{b}_1$;
- $f(\mathbf{b}_2 + \mathbf{b}_3) = \mathbf{b}_2$;
- $f(\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3) = \mathbf{b}_3$.

(e) Let f be as in part (d). Compute the following matrices:

(1) ${}_{\mathcal{C}}\left[\ f\ \right]_{\mathcal{B}}$; *(2)* ${}_{\mathcal{B}}\left[\ f\ \right]_{\mathcal{C}}$; *(3)* ${}_{\mathcal{B}}\left[\ f\ \right]_{\mathcal{B}}$.

You should be able to just "read off" one of these matrices (which one?) from the specifications given in part (d). Then, you can compute the other two using the change of bases matrices that you computed in part (b).

(f) Once again, let f be as in part (d). Using coordinate vectors and your answer to part (e), compute $f(\mathbf{b}_1)$.

Exercise 2. *Let U and V be vector spaces over $\mathbb{Z}_3$, let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis of U, and let $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4\}$ be a basis of V. Consider the unique linear function $f : U \to V$ that satisfies the following:*

- *$f(\mathbf{b}_1) = \mathbf{c}_1 + \mathbf{c}_2$;*
- *$f(\mathbf{b}_2) = \mathbf{c}_2 + 2\mathbf{c}_3 + 2\mathbf{c}_4$;*
- *$f(\mathbf{b}_3) = \mathbf{c}_2 + \mathbf{c}_3$.*

(Why does f exist and why is it unique?) Compute the matrix ${}_{\mathcal{C}}[\ f\ ]_{\mathcal{B}}$ and compute rank(f). Is f one-to-one? Is f onto? Is f an isomorphism?

Exercise 3.

(a) *Prove that there exists a linear function $f : \mathbb{Z}_2^{2\times 2} \to \mathbb{Z}_2^3$ that satisfies the following properties:*

- $f(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$;
- $f(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$;
- $f(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$.

(b) *Is the linear function f from part (a) unique?*

(c) *Can a linear function f satisfying the properties from part (a) be one-to-one? Can it be onto?*

(d) *Find a formula for some linear function f satisfying the properties from part (a). Can you find more than one correct answer? Can you come up with examples of different rank?*