Linear Algebra 2

Lecture #22

The Cayley-Hamilton theorem. Diagonalization

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  1. The Cayley-Hamilton theorem

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    2. Eigenvectors and linear independence. Eigenbases

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  3. Diagonalization

1. The Cayley-Hamilton theorem

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The Cayley-Hamilton theorem

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times n}$ be a square matrix, and let $p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ be the characteristic polynomial of A. Then

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n \quad = \quad O_{n \times n}.$$

- The Cayley-Hamilton theorem essentially states that every square matrix is a root of its own characteristic polynomial.
  - Here, we need to treat the free coefficient of the characteristic polynomial as that coefficient times the identity matrix of the appropriate size.

- For example, for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$ or $\mathbb{C}$, we have that

$$p_A(\lambda) = \det(\lambda I_2 - A) = \begin{vmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{vmatrix} = \lambda^2 - 5\lambda - 2,$$

and we see that

$$\begin{aligned} A^2 - 5A - 2I_2 &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 - 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

The Cayley-Hamilton theorem

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times n}$ be a square matrix, and let $p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ be the characteristic polynomial of A. Then

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- The proof of the Cayley-Hamilton theorem relies on the adjugate matrix and on the theorem below.

Theorem 7.8.2

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$ $(n \geq 2)$. Then

$$\text{adj}(A)\, A \quad = \quad A\, \text{adj}(A) \quad = \quad \det(A) I_n.$$

Consequently, if A is invertible, then $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$.

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Proof.

The Cayley-Hamilton theorem

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$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n \quad = \quad O_{n\times n}.$$

Proof. If $n = 1$, then the result is immediate.

- Indeed, suppose that $n = 1$, and consider any matrix $A = \left[\ a_{1,1}\ \right]$ in $\mathbb{F}^{1\times 1}$.
- Then $p_A(\lambda) = \det(\lambda I_1 - A) = \det(\left[\ \lambda - a_{1,1}\ \right]) = \lambda - a_{1,1}$, and we see that $A - a_{1,1}I_1 = O_{1\times 1}$.

Proof (continued). From now on, we assume that $n \geq 2$.

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$$(\lambda I_n - A)\,\mathrm{adj}(\lambda I_n - A) \;\; = \;\; \det(\lambda I_n - A) I_n.$$

Proof (continued). From now on, we assume that $n \geq 2$. By Theorem 7.8.2 applied to the matrix $\lambda I_n - A$ (where λ is a variable), we get that

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Now, note that each cofactor of the matrix $\lambda I_n - A$ is a polynomial (in variable λ) of degree at most λ^{n-1}.

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Now, note that each cofactor of the matrix $\lambda I_n - A$ is a polynomial (in variable λ) of degree at most λ^{n-1}. Since the entries of $\mathrm{adj}(\lambda I_n - A)$ are precisely the cofactors of $\lambda I_n - A$, it follows that each entry of $\mathrm{adj}(\lambda I_n - A)$ is a polynomial (in the variable λ) of degree at most $n - 1$.

Proof (continued). From now on, we assume that $n \geq 2$. By Theorem 7.8.2 applied to the matrix $\lambda I_n - A$ (where λ is a variable), we get that

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$$\text{adj}(\lambda I_n - A) \;\; = \;\; \lambda^{n-1} B_{n-1} + \lambda^{n-2} B_{n-2} + \cdots + \lambda B_1 + B_0,$$

for some matrices $B_0, B_1, \dots, B_{n-1} \in \mathbb{F}^{n \times n}$.

Proof (continued). From now on, we assume that $n \geq 2$. By Theorem 7.8.2 applied to the matrix $\lambda I_n - A$ (where λ is a variable), we get that

$$(\lambda I_n - A)\,\text{adj}(\lambda I_n - A) \;\;=\;\; \det(\lambda I_n - A) I_n.$$

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for some matrices $B_0, B_1, \dots, B_{n-1} \in \mathbb{F}^{n \times n}$. Consequently,

$$\underbrace{(\lambda I_n - A)(\underbrace{\lambda^{n-1} B_{n-1} + \lambda^{n-2} B_{n-2} + \dots + \lambda B_1 + B_0}_{=\text{adj}(\lambda I_n - A)})}_{:=\text{LHS}} \;\;=\;\; \underbrace{\det(\lambda I_n - A) I_n}_{:=\text{RHS}}.$$

Proof (continued). Reminder: $n \geq 2$,

$$\underbrace{(\lambda I_n - A)(\underbrace{\lambda^{n-1}B_{n-1} + \lambda^{n-2}B_{n-2} + \cdots + \lambda B_1 + B_0}_{=\mathrm{adj}(\lambda I_n - A)})}_{:=\mathrm{LHS}} = \underbrace{\det(\lambda I_n - A)I_n}_{:=\mathrm{RHS}}.$$

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For the left-hand-side, we have

$$\begin{aligned} \mathrm{LHS} &= (\lambda I_n - A)(\lambda^{n-1}B_{n-1} + \cdots + \lambda B_1 + B_0) \\ &= \lambda^n B_{n-1} + \lambda^{n-1}(B_{n-2} - AB_{n-1}) + \lambda^{n-2}(B_{n-3} - AB_{n-2}) + \\ &\quad + \cdots + \lambda(B_0 - AB_1) - AB_0. \end{aligned}$$

Proof (continued). Reminder: $n \geq 2$,

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For the right-hand-side, we have

$$\begin{aligned} \text{RHS} &= \det(\lambda I_n - A)I_n = p_A(\lambda)I_n \\ &= (\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-1} + \cdots + a_1\lambda + a_0)I_n \\ &= \lambda^n I_n + \lambda^{n-1}a_{n-1}I_n + \lambda^{n-2}a_{n-2}I_n + \cdots + \lambda a_1 I_n + a_0 I_n. \end{aligned}$$

Proof (continued). Reminder: $n \geq 2$,

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The corresponding coefficients in front of λ^i (for $i \in \{0, 1, \ldots, n\}$) must be equal on the left-hand-side (LHS) and the right-hand-side (RHS).

Proof (continued). Reminder: $n \geq 2$,

$$\underbrace{(\lambda I_n - A)(\underbrace{\lambda^{n-1}B_{n-1} + \lambda^{n-2}B_{n-2} + \cdots + \lambda B_1 + B_0}_{=\mathrm{adj}(\lambda I_n - A)})}_{:=\mathrm{LHS}} = \underbrace{\det(\lambda I_n - A) I_n}_{:=\mathrm{RHS}}.$$

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For the right-hand-side, we have

$$\begin{aligned}\mathrm{RHS} &= \det(\lambda I_n - A) I_n = p_A(\lambda) I_n \\ &= (\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-1} + \cdots + a_1\lambda + a_0) I_n \\ &= \lambda^n I_n + \lambda^{n-1} a_{n-1} I_n + \lambda^{n-2} a_{n-2} I_n + \cdots + \lambda a_1 I_n + a_0 I_n.\end{aligned}$$

The corresponding coefficients in front of λ^i (for $i \in \{0, 1, \ldots, n\}$) must be equal on the left-hand-side (LHS) and the right-hand-side (RHS). This yields the following $n+1$ equations (next slide).

Proof (continued).

$$\begin{aligned}
B_{n-1} &= I_n \\
B_{n-2} - AB_{n-1} &= a_{n-1} I_n \\
B_{n-3} - AB_{n-2} &= a_{n-2} I_n \\
&\vdots \\
B_0 - AB_1 &= a_1 I_n \\
-AB_0 &= a_0 I_n
\end{aligned}$$

Proof (continued).

$$\begin{array}{rcl} B_{n-1} & = & I_n \\ B_{n-2} - AB_{n-1} & = & a_{n-1} I_n \\ B_{n-3} - AB_{n-2} & = & a_{n-2} I_n \\ & \vdots & \\ B_0 - AB_1 & = & a_1 I_n \\ -AB_0 & = & a_0 I_n \end{array}$$

We now multiply the first (top) equation by A^n on the left, the second equation by A^{n-1} on the left, the third equation by A^{n-2} on the left, and so on.

Proof (continued).

$$\begin{aligned} B_{n-1} &= I_n \\ B_{n-2} - AB_{n-1} &= a_{n-1} I_n \\ B_{n-3} - AB_{n-2} &= a_{n-2} I_n \\ &\vdots \\ B_0 - AB_1 &= a_1 I_n \\ -AB_0 &= a_0 I_n \end{aligned}$$

We now multiply the first (top) equation by A^n on the left, the second equation by A^{n-1} on the left, the third equation by A^{n-2} on the left, and so on. This yields the following.

$$\begin{aligned} A^n B_{n-1} &= A^n \\ A^{n-1} B_{n-2} - A^n B_{n-1} &= a_{n-1} A^{n-1} \\ A^{n-2} B_{n-3} - A^{n-1} B_{n-2} &= a_{n-2} A^{n-2} \\ &\vdots \\ AB_0 - A^2 B_1 &= a_1 A \\ -AB_0 &= a_0 I_n \end{aligned}$$

Proof (continued). Reminder:

$$
\begin{aligned}
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We now add up the equations that we obtained.

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We now add up the equations that we obtained.

On the left-hand-side, the sum is obviously $O_{n \times n}$.

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We now add up the equations that we obtained.

On the left-hand-side, the sum is obviously $O_{n \times n}$.

So, the right-hand-side must also sum up to $O_{n \times n}$, i.e.

$$A^n + a_{n-1} A^{n-1} + a_{n-2} A^{n-2} + \cdots + a_1 A + a_0 I_n = O_{n \times n}.$$

But this is precisely what we needed to show. □

The Cayley-Hamilton theorem

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times n}$ be a square matrix, and let $p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ be the characteristic polynomial of A. Then

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Corollary 8.3.1

Let $\mathbb{F}$ be a field. For all matrices $A \in \mathbb{F}^{n\times n}$:

(a) $A^n \in \text{Span}(I_n, A, A^2, \dots, A^{n-1})$, i.e. A^n is a linear combination of $I_n, A, A^2, \dots, A^{n-1}$;

(b) if A is invertible, then $A^{-1} \in \text{Span}(I_n, A, A^2, \dots, A^{n-1})$, i.e. A^{-1} is a linear combination of $I_n, A, A^2, \dots, A^{n-1}$.

Proof.

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Let $\mathbb{F}$ be a field. For all matrices $A \in \mathbb{F}^{n \times n}$:

- (a) $A^n \in \text{Span}(I_n, A, A^2, \dots, A^{n-1})$, i.e. A^n is a linear combination of $I_n, A, A^2, \dots, A^{n-1}$;
- (b) if A is invertible, then $A^{-1} \in \text{Span}(I_n, A, A^2, \dots, A^{n-1})$, i.e. A^{-1} is a linear combination of $I_n, A, A^2, \dots, A^{n-1}$.

Proof. Fix a matrix $A \in \mathbb{F}^{n \times n}$, and consider its characteristic polynomial $p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0$.

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Proof of (a).

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Proof of (a). By the Cayley-Hamilton theorem, we have that

$$A^n + a_{n-1}A^{n-1} + \cdots + a_a A^2 + a_1 A + a_0 I_n \quad = \quad O_{n\times n}.$$

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Consequently,

$$A^n \;=\; -a_0 I_n - a_1 A - a_2 A^2 - \cdots - a_{n-1}A^{n-1}.$$

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Consequently,

$$A^n \quad = \quad -a_0 I_n - a_1 A - a_2 A^2 - \cdots - a_{n-1}A^{n-1}.$$

Thus, A^n is a linear combination of the matrices $I_n, A, A^2, \ldots, A^{n-1}$.

Corollary 8.3.1

(b) if A is invertible, then $A^{-1} \in \text{Span}(I_n, A, A^2, \ldots, A^{n-1})$, i.e. A^{-1} is a linear combination of $I_n, A, A^2, \ldots, A^{n-1}$.

- Reminder:
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Proof of (b). Assume that A is invertible.

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Proof of (b). Assume that A is invertible. Proposition 8.2.11 then guarantees that 0 is not an eigenvalue of A.

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- Reminder:
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Proof of (b). Assume that A is invertible. Proposition 8.2.11 then guarantees that 0 is not an eigenvalue of A. Since the eigenvalues of A are precisely the roots of the characteristic polynomial of A, we have that $p_A(0) \neq 0$; since $p_A(0) = a_0$, it follows that $a_0 \neq 0$.

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Proof of (b). Assume that A is invertible. Proposition 8.2.11 then guarantees that 0 is not an eigenvalue of A. Since the eigenvalues of A are precisely the roots of the characteristic polynomial of A, we have that $p_A(0) \neq 0$; since $p_A(0) = a_0$, it follows that $a_0 \neq 0$.

Now, by the Cayley-Hamilton theorem, we have that

$$A^n + a_{n-1}A^{n-1} + \cdots + a_2A^2 + a_1A + a_0I_n \;=\; O_{n\times n}.$$

Corollary 8.3.1

(b) if A is invertible, then $A^{-1} \in \mathrm{Span}(I_n, A, A^2, \ldots, A^{n-1})$, i.e. A^{-1} is a linear combination of $I_n, A, A^2, \ldots, A^{n-1}$.

Proof of (b) (continued). Reminder: $a_0 \neq 0$,

$$A^n + a_{n-1}A^{n-1} + \cdots + a_2A^2 + a_1A + a_0I_n \quad = \quad O_{n\times n}.$$

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Proof of (b) (continued). Reminder: $a_0 \neq 0$,

$$A^n + a_{n-1}A^{n-1} + \cdots + a_2A^2 + a_1A + a_0I_n \quad = \quad O_{n\times n}.$$

We multiply both sides of the equation by A^{-1} on the right, and we obtain

$$A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_2A + a_1I_n + a_0A^{-1} \quad = \quad O_{n\times n},$$

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and consequently,

$$a_0A^{-1} = -a_1I_n - a_2A - \cdots - a_{n-1}A^{n-2} - A^{n-1}.$$

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(b) if A is invertible, then $A^{-1} \in \text{Span}(I_n, A, A^2, \dots, A^{n-1})$, i.e. A^{-1} is a linear combination of $I_n, A, A^2, \dots, A^{n-1}$.

Proof of (b) (continued). Reminder: $a_0 \neq 0$,

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Since $a_0 \neq 0$, this implies that

$$A^{-1} \quad = \quad -\tfrac{a_1}{a_0}I_n - \tfrac{a_2}{a_0}A - \cdots - \tfrac{a_{n-1}}{a_0}A^{n-2} - \tfrac{1}{a_0}A^{n-1}.$$

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(b) if A is invertible, then $A^{-1} \in \text{Span}(I_n, A, A^2, \dots, A^{n-1})$, i.e. A^{-1} is a linear combination of $I_n, A, A^2, \dots, A^{n-1}$.

Proof of (b) (continued). Reminder: $a_0 \neq 0$,

$$A^n + a_{n-1}A^{n-1} + \cdots + a_2A^2 + a_1A + a_0I_n = O_{n\times n}.$$

We multiply both sides of the equation by A^{-1} on the right, and we obtain

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$$A^{-1} = -\frac{a_1}{a_0}I_n - \frac{a_2}{a_0}A - \cdots - \frac{a_{n-1}}{a_0}A^{n-2} - \frac{1}{a_0}A^{n-1}.$$

So, A^{-1} is a linear combination of $I_n, A, A^2, \dots, A^{n-1}$. $\square$

The Cayley-Hamilton theorem

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times n}$ be a square matrix, and let $p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ be the characteristic polynomial of A. Then

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n \quad = \quad O_{n\times n}.$$

Corollary 8.3.1

Let $\mathbb{F}$ be a field. For all matrices $A \in \mathbb{F}^{n\times n}$:

- (a) $A^n \in \text{Span}(I_n, A, A^2, \dots, A^{n-1})$, i.e. A^n is a linear combination of $I_n, A, A^2, \dots, A^{n-1}$;
- (b) if A is invertible, then $A^{-1} \in \text{Span}(I_n, A, A^2, \dots, A^{n-1})$, i.e. A^{-1} is a linear combination of $I_n, A, A^2, \dots, A^{n-1}$.

2. Eigenvectors and linear independence. Eigenbases

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Definition

For a finite-dimensional vector space V over a field $\mathbb{F}$ and a linear function $f : V \to V$, an *eigenbasis* of V associated with f is a basis $\mathcal{B}$ of V s.t. all vectors in $\mathcal{B}$ are eigenvectors of f.

Definition

For an field $\mathbb{F}$ and a matrix $A \in \mathbb{F}^{n \times n}$, an *eigenbasis* of $\mathbb{F}^n$ associated with A is a basis $\mathcal{B}$ of $\mathbb{F}^n$ s.t. all vectors in $\mathcal{B}$ are eigenvectors of A.

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- Eigenbases do not always exist, and one of our goals in this section is to determine when they do and do not exist.
- As we shall see (later!), eigenbases play a crucial role in matrix "diagonalization."

Proposition 8.4.1

Let V be a vector space over a field $\mathbb{F}$, let $f : V \to V$ be a linear function, let $\lambda_1, \ldots, \lambda_k \in \mathbb{F}$ be pairwise distinct eigenvalues of f, associated with eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$, respectively. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent set.

Proof.

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Proof. We will prove inductively that for all $i \in \{0, \ldots, k\}$, the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$ is linearly independent.

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For $i = 0$, we have that $\{\mathbf{v}_1, \dots, \mathbf{v}_i\} = \emptyset$, which is obviously a linearly independent set.

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Now, fix an index $i \in \{0, \dots, k-1\}$, and assume inductively that the set $\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$ is linearly independent. We must show that $\{\mathbf{v}_1, \dots, \mathbf{v}_i, \mathbf{v}_{i+1}\}$ is linearly independent.

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Now, fix an index $i \in \{0, \dots, k-1\}$, and assume inductively that the set $\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$ is linearly independent. We must show that $\{\mathbf{v}_1, \dots, \mathbf{v}_i, \mathbf{v}_{i+1}\}$ is linearly independent. Fix scalars $\alpha_1, \dots, \alpha_i, \alpha_{i+1} \in \mathbb{F}$ s.t.

$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_i \mathbf{v}_i + \alpha_{i+1} \mathbf{v}_{i+1} = \mathbf{0}.$$

WTS $\alpha_1 = \dots = \alpha_i = \alpha_{i+1} = 0$.

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Proof (continued). Reminder: $\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$ is linearly independent; $\alpha_1\mathbf{v}_1 + \dots + \alpha_i\mathbf{v}_i + \alpha_{i+1}\mathbf{v}_{i+1} = \mathbf{0}$; WTS $\alpha_1 = \dots = \alpha_i = \alpha_{i+1} = 0$.

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If we multiply both sides of the equation above by λ_{i+1}, we obtain

1. $\lambda_{i+1}\alpha_1\mathbf{v}_1 + \dots + \lambda_{i+1}\alpha_i\mathbf{v}_i + \lambda_{i+1}\alpha_{i+1}\mathbf{v}_{i+1} = \mathbf{0}$.

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Proof (continued). Reminder: $\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$ is linearly independent; $\alpha_1 \mathbf{v}_1 + \dots + \alpha_i \mathbf{v}_i + \alpha_{i+1} \mathbf{v}_{i+1} = \mathbf{0}$; WTS $\alpha_1 = \dots = \alpha_i = \alpha_{i+1} = 0$.

If we multiply both sides of the equation above by λ_{i+1}, we obtain

1. $\lambda_{i+1}\alpha_1 \mathbf{v}_1 + \dots + \lambda_{i+1}\alpha_i \mathbf{v}_i + \lambda_{i+1}\alpha_{i+1} \mathbf{v}_{i+1} = \mathbf{0}$.

If, on the other hand, we apply the function f to both sides and also use the fact that $f(\mathbf{0}) = \mathbf{0}$, then we obtain

2. $f\big(\alpha_1 \mathbf{v}_1 + \dots + \alpha_i \mathbf{v}_i + \alpha_{i+1} \mathbf{v}_{i+1}\big) = \mathbf{0}$.

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Let V be a vector space over a field $\mathbb{F}$, let $f : V \to V$ be a linear function, let $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ be pairwise distinct eigenvalues of f, associated with eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_k$, respectively. Then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent set.

Proof (continued). We now compute:

$$\begin{aligned} \mathbf{0} &\stackrel{(2)}{=} f(\alpha_1\mathbf{v}_1 + \dots + \alpha_i\mathbf{v}_i + \alpha_{i+1}\mathbf{v}_{i+1}) \\ &\stackrel{(*)}{=} \alpha_1 f(\mathbf{v}_1) + \dots + \alpha_i f(\mathbf{v}_i) + \alpha_{i+1} f(\mathbf{v}_{i+1}) \\ &\stackrel{(**)}{=} \alpha_1\lambda_1\mathbf{v}_1 + \dots + \alpha_i\lambda_i\mathbf{v}_i + \alpha_{i+1}\lambda_{i+1}\mathbf{v}_{i+1}, \end{aligned}$$

where (*) follows from the linearity of f, and (**) follows from the fact that $\mathbf{v}_1, \dots, \mathbf{v}_i, \mathbf{v}_{i+1}$ are eigenvectors of f associated with eigenvalues $\lambda_1, \dots, \lambda_i, \lambda_{i+1}$, respectively.

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Let V be a vector space over a field $\mathbb{F}$, let $f : V \to V$ be a linear function, let $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ be pairwise distinct eigenvalues of f, associated with eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_k$, respectively. Then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent set.

Proof (continued). Reminder:

1. $\lambda_{i+1}\alpha_1\mathbf{v}_1 + \dots + \lambda_{i+1}\alpha_i\mathbf{v}_i + \lambda_{i+1}\alpha_{i+1}\mathbf{v}_{i+1} = \mathbf{0}$;
2. $f\big(\alpha_1\mathbf{v}_1 + \dots + \alpha_i\mathbf{v}_i + \alpha_{i+1}\mathbf{v}_{i+1}\big) = \mathbf{0}$;
3. $\alpha_1\lambda_1\mathbf{v}_1 + \dots + \alpha_i\lambda_i\mathbf{v}_i + \alpha_{i+1}\lambda_{i+1}\mathbf{v}_{i+1} = \mathbf{0}$.

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Proof (continued). Reminder:

1. $\lambda_{i+1}\alpha_1\mathbf{v}_1 + \dots + \lambda_{i+1}\alpha_i\mathbf{v}_i + \lambda_{i+1}\alpha_{i+1}\mathbf{v}_{i+1} = \mathbf{0}$;
2. $f\big(\alpha_1\mathbf{v}_1 + \dots + \alpha_i\mathbf{v}_i + \alpha_{i+1}\mathbf{v}_{i+1}\big) = \mathbf{0}$;
3. $\alpha_1\lambda_1\mathbf{v}_1 + \dots + \alpha_i\lambda_i\mathbf{v}_i + \alpha_{i+1}\lambda_{i+1}\mathbf{v}_{i+1} = \mathbf{0}$.

Combining (1) and (3), we obtain:

$$\begin{aligned} & \alpha_1\lambda_1\mathbf{v}_1 + \dots + \alpha_i\lambda_i\mathbf{v}_i + \alpha_{i+1}\lambda_{i+1}\mathbf{v}_{i+1} \\ = \; & \lambda_{i+1}\alpha_1\mathbf{v}_1 + \dots + \lambda_{i+1}\alpha_i\mathbf{v}_i + \lambda_{i+1}\alpha_{i+1}\mathbf{v}_{i+1}. \end{aligned}$$

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Let V be a vector space over a field $\mathbb{F}$, let $f : V \to V$ be a linear function, let $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ be pairwise distinct eigenvalues of f, associated with eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_k$, respectively. Then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent set.

Proof (continued). Reminder:

1. $\lambda_{i+1}\alpha_1\mathbf{v}_1 + \dots + \lambda_{i+1}\alpha_i\mathbf{v}_i + \lambda_{i+1}\alpha_{i+1}\mathbf{v}_{i+1} = \mathbf{0}$;
2. $f(\alpha_1\mathbf{v}_1 + \dots + \alpha_i\mathbf{v}_i + \alpha_{i+1}\mathbf{v}_{i+1}) = \mathbf{0}$;
3. $\alpha_1\lambda_1\mathbf{v}_1 + \dots + \alpha_i\lambda_i\mathbf{v}_i + \alpha_{i+1}\lambda_{i+1}\mathbf{v}_{i+1} = \mathbf{0}$.

Combining (1) and (3), we obtain:

$$\begin{aligned} & \alpha_1\lambda_1\mathbf{v}_1 + \dots + \alpha_i\lambda_i\mathbf{v}_i + \alpha_{i+1}\lambda_{i+1}\mathbf{v}_{i+1} \\ = \quad & \lambda_{i+1}\alpha_1\mathbf{v}_1 + \dots + \lambda_{i+1}\alpha_i\mathbf{v}_i + \lambda_{i+1}\alpha_{i+1}\mathbf{v}_{i+1}. \end{aligned}$$

By subtracting one side from the other and factoring, we get

$$\alpha_1(\lambda_1 - \lambda_{i+1})\mathbf{v}_1 + \dots + \alpha_i(\lambda_i - \lambda_{i+1})\mathbf{v}_i \quad = \quad \mathbf{0}.$$

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Let V be a vector space over a field $\mathbb{F}$, let $f : V \to V$ be a linear function, let $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ be pairwise distinct eigenvalues of f, associated with eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_k$, respectively. Then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent set.

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Proof (continued). Reminder:

$$\alpha_1(\lambda_1 - \lambda_{i+1})\mathbf{v}_1 + \dots + \alpha_i(\lambda_i - \lambda_{i+1})\mathbf{v}_i \quad = \quad \mathbf{0}$$

By the ind. hyp., $\mathbf{v}_1, \dots, \mathbf{v}_i$ are linearly independent, and it follows that $\alpha_1(\lambda_1 - \lambda_{i+1}) = \dots = \alpha_i(\lambda_i - \lambda_{i+1}) = 0$.

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Proof (continued). Reminder:

$$\alpha_1(\lambda_1 - \lambda_{i+1})\mathbf{v}_1 + \dots + \alpha_i(\lambda_i - \lambda_{i+1})\mathbf{v}_i = \mathbf{0}$$

By the ind. hyp., $\mathbf{v}_1, \dots, \mathbf{v}_i$ are linearly independent, and it follows that $\alpha_1(\lambda_1 - \lambda_{i+1}) = \dots = \alpha_i(\lambda_i - \lambda_{i+1}) = 0$. Since $\lambda_1 - \lambda_{i+1}, \dots, \lambda_i - \lambda_{i+1}$ are all non-zero (because $\lambda_1, \dots, \lambda_i, \lambda_{i+1}$ are pairwise distinct), we deduce that $\alpha_1 = \dots = \alpha_i = 0$.

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Proof (continued). Reminder:

$$\alpha_1(\lambda_1 - \lambda_{i+1})\mathbf{v}_1 + \dots + \alpha_i(\lambda_i - \lambda_{i+1})\mathbf{v}_i \quad = \quad \mathbf{0}$$

By the ind. hyp., $\mathbf{v}_1, \dots, \mathbf{v}_i$ are linearly independent, and it follows that $\alpha_1(\lambda_1 - \lambda_{i+1}) = \dots = \alpha_i(\lambda_i - \lambda_{i+1}) = 0$. Since $\lambda_1 - \lambda_{i+1}, \dots, \lambda_i - \lambda_{i+1}$ are all non-zero (because $\lambda_1, \dots, \lambda_i, \lambda_{i+1}$ are pairwise distinct), we deduce that $\alpha_1 = \dots = \alpha_i = 0$. Plugging this into our equation $\alpha_1\mathbf{v}_1 + \dots + \alpha_i\mathbf{v}_i + \alpha_{i+1}\mathbf{v}_{i+1} = \mathbf{0}$, we get

$$\alpha_{i+1}\mathbf{v}_{i+1} \quad = \quad \mathbf{0}.$$

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$$\alpha_1(\lambda_1 - \lambda_{i+1})\mathbf{v}_1 + \dots + \alpha_i(\lambda_i - \lambda_{i+1})\mathbf{v}_i \quad = \quad \mathbf{0}$$

By the ind. hyp., $\mathbf{v}_1, \dots, \mathbf{v}_i$ are linearly independent, and it follows that $\alpha_1(\lambda_1 - \lambda_{i+1}) = \dots = \alpha_i(\lambda_i - \lambda_{i+1}) = 0$. Since $\lambda_1 - \lambda_{i+1}, \dots, \lambda_i - \lambda_{i+1}$ are all non-zero (because $\lambda_1, \dots, \lambda_i, \lambda_{i+1}$ are pairwise distinct), we deduce that $\alpha_1 = \dots = \alpha_i = 0$. Plugging this into our equation $\alpha_1\mathbf{v}_1 + \dots + \alpha_i\mathbf{v}_i + \alpha_{i+1}\mathbf{v}_{i+1} = \mathbf{0}$, we get

$$\alpha_{i+1}\mathbf{v}_{i+1} \quad = \quad \mathbf{0}.$$

But $\mathbf{v}_{i+1}$ is an eigenvector of f, and so by definition, $\mathbf{v}_{i+1} \neq \mathbf{0}$.

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Let V be a vector space over a field $\mathbb{F}$, let $f : V \to V$ be a linear function, let $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ be pairwise distinct eigenvalues of f, associated with eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_k$, respectively. Then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent set.

Proof (continued). Reminder:

$$\alpha_1(\lambda_1 - \lambda_{i+1})\mathbf{v}_1 + \dots + \alpha_i(\lambda_i - \lambda_{i+1})\mathbf{v}_i \quad = \quad \mathbf{0}$$

By the ind. hyp., $\mathbf{v}_1, \dots, \mathbf{v}_i$ are linearly independent, and it follows that $\alpha_1(\lambda_1 - \lambda_{i+1}) = \dots = \alpha_i(\lambda_i - \lambda_{i+1}) = 0$. Since $\lambda_1 - \lambda_{i+1}, \dots, \lambda_i - \lambda_{i+1}$ are all non-zero (because $\lambda_1, \dots, \lambda_i, \lambda_{i+1}$ are pairwise distinct), we deduce that $\alpha_1 = \dots = \alpha_i = 0$. Plugging this into our equation $\alpha_1\mathbf{v}_1 + \dots + \alpha_i\mathbf{v}_i + \alpha_{i+1}\mathbf{v}_{i+1} = \mathbf{0}$, we get

$$\alpha_{i+1}\mathbf{v}_{i+1} \quad = \quad \mathbf{0}.$$

But $\mathbf{v}_{i+1}$ is an eigenvector of f, and so by definition, $\mathbf{v}_{i+1} \neq \mathbf{0}$. So, $\alpha_{i+1} = 0$. Thus, $\alpha_1 = \dots = \alpha_i = \alpha_{i+1} = 0$.

Proposition 8.4.1

Let V be a vector space over a field $\mathbb{F}$, let $f : V \to V$ be a linear function, let $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ be pairwise distinct eigenvalues of f, associated with eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_k$, respectively. Then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent set.

Proof (continued). Reminder:

$$\alpha_1(\lambda_1 - \lambda_{i+1})\mathbf{v}_1 + \dots + \alpha_i(\lambda_i - \lambda_{i+1})\mathbf{v}_i \quad = \quad \mathbf{0}$$

By the ind. hyp., $\mathbf{v}_1, \dots, \mathbf{v}_i$ are linearly independent, and it follows that $\alpha_1(\lambda_1 - \lambda_{i+1}) = \dots = \alpha_i(\lambda_i - \lambda_{i+1}) = 0$. Since $\lambda_1 - \lambda_{i+1}, \dots, \lambda_i - \lambda_{i+1}$ are all non-zero (because $\lambda_1, \dots, \lambda_i, \lambda_{i+1}$ are pairwise distinct), we deduce that $\alpha_1 = \dots = \alpha_i = 0$. Plugging this into our equation $\alpha_1\mathbf{v}_1 + \dots + \alpha_i\mathbf{v}_i + \alpha_{i+1}\mathbf{v}_{i+1} = \mathbf{0}$, we get

$$\alpha_{i+1}\mathbf{v}_{i+1} \quad = \quad \mathbf{0}.$$

But $\mathbf{v}_{i+1}$ is an eigenvector of f, and so by definition, $\mathbf{v}_{i+1} \neq \mathbf{0}$. So, $\alpha_{i+1} = 0$. Thus, $\alpha_1 = \dots = \alpha_i = \alpha_{i+1} = 0$. So, $\{\mathbf{v}_1, \dots, \mathbf{v}_i, \mathbf{v}_{i+1}\}$ is linearly independent. This completes the induction. $\square$

Proposition 8.4.1

Let V be a vector space over a field $\mathbb{F}$, let $f : V \to V$ be a linear function, let $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ be pairwise distinct eigenvalues of f, associated with eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_k$, respectively. Then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent set.

Proposition 8.4.1

Let V be a vector space over a field $\mathbb{F}$, let $f : V \to V$ be a linear function, let $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ be pairwise distinct eigenvalues of f, associated with eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_k$, respectively. Then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent set.

Proposition 8.4.2

Let V be a vector space over a field $\mathbb{F}$, let $f : V \to V$ be a linear function, and let $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{F}$ be pairwise distinct eigenvalues of f. For each $i \in \{1, \dots, k\}$, let $\mathbf{v}_{i,1}, \dots, \mathbf{v}_{i,t_i}$ be linearly independent eigenvectors of f associated with the eigenvalue λ_i. Then the eigenvectors

$$\mathbf{v}_{1,1}, \dots, \mathbf{v}_{1,t_1}, \mathbf{v}_{2,1}, \dots, \mathbf{v}_{2,t_2}, \dots, \mathbf{v}_{k,1}, \dots, \mathbf{v}_{k,t_k}$$

are linearly independent.

Proof.

Proof. Fix $\alpha_{1,1}, \ldots, \alpha_{1,t_1}, \alpha_{2,1}, \ldots, \alpha_{2,t_2}, \ldots, \alpha_{k,1}, \ldots, \alpha_{k,t_k} \in \mathbb{F}$ s.t.

$$\sum_{i=1}^{k} \left(\alpha_{i,1} \mathbf{v}_{i,1} + \cdots + \alpha_{i,t_i} \mathbf{v}_{i,t_i} \right) = \mathbf{0}.$$

Proof. Fix $\alpha_{1,1}, \ldots, \alpha_{1,t_1}, \alpha_{2,1}, \ldots, \alpha_{2,t_2}, \ldots, \alpha_{k,1}, \ldots, \alpha_{k,t_k} \in \mathbb{F}$ s.t.

$$\sum_{i=1}^{k} \left(\alpha_{i,1}\mathbf{v}_{i,1} + \cdots + \alpha_{i,t_i}\mathbf{v}_{i,t_i}\right) = \mathbf{0}.$$

$\forall i \in \{1, \ldots, k\}$: set $\mathbf{v}_i := \alpha_{i,1}\mathbf{v}_{i,1} + \cdots + \alpha_{i,t_i}\mathbf{v}_{i,t_i}$, that is

- $\mathbf{v}_1 := \alpha_{1,1}\mathbf{v}_{1,1} + \cdots + \alpha_{1,t_1}\mathbf{v}_{1,t_1}$;
- $\mathbf{v}_2 := \alpha_{2,1}\mathbf{v}_{2,1} + \cdots + \alpha_{2,t_2}\mathbf{v}_{2,t_2}$;

  $\vdots$
- $\mathbf{v}_k := \alpha_{k,1}\mathbf{v}_{k,1} + \cdots + \alpha_{k,t_k}\mathbf{v}_{k,t_k}$.

Proof. Fix $\alpha_{1,1}, \dots, \alpha_{1,t_1}, \alpha_{2,1}, \dots, \alpha_{2,t_2}, \dots, \alpha_{k,1}, \dots, \alpha_{k,t_k} \in \mathbb{F}$ s.t.

$$\sum_{i=1}^{k} \left(\alpha_{i,1}\mathbf{v}_{i,1} + \dots + \alpha_{i,t_i}\mathbf{v}_{i,t_i} \right) = \mathbf{0}.$$

$\forall i \in \{1, \dots, k\}$: set $\mathbf{v}_i := \alpha_{i,1}\mathbf{v}_{i,1} + \dots + \alpha_{i,t_i}\mathbf{v}_{i,t_i}$, that is

- $\mathbf{v}_1 := \alpha_{1,1}\mathbf{v}_{1,1} + \dots + \alpha_{1,t_1}\mathbf{v}_{1,t_1}$;
- $\mathbf{v}_2 := \alpha_{2,1}\mathbf{v}_{2,1} + \dots + \alpha_{2,t_2}\mathbf{v}_{2,t_2}$;

  $\vdots$

- $\mathbf{v}_k := \alpha_{k,1}\mathbf{v}_{k,1} + \dots + \alpha_{k,t_k}\mathbf{v}_{k,t_k}$.

So, $\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_k = \mathbf{0}$. Now, note that for each $i \in \{1, \dots, k\}$, the vector $\mathbf{v}_i$ is a linear combination of vectors in $E_{\lambda_i}(f)$;

Proof. Fix $\alpha_{1,1},\dots,\alpha_{1,t_1},\alpha_{2,1},\dots,\alpha_{2,t_2},\dots,\alpha_{k,1},\dots,\alpha_{k,t_k}\in\mathbb{F}$ s.t.

$$\sum_{i=1}^{k}\Big(\alpha_{i,1}\mathbf{v}_{i,1}+\cdots+\alpha_{i,t_i}\mathbf{v}_{i,t_i}\Big) \quad = \quad \mathbf{0}.$$

$\forall i\in\{1,\dots,k\}$: set $\mathbf{v}_i := \alpha_{i,1}\mathbf{v}_{i,1}+\cdots+\alpha_{i,t_i}\mathbf{v}_{i,t_i}$, that is

- $\mathbf{v}_1 := \alpha_{1,1}\mathbf{v}_{1,1}+\cdots+\alpha_{1,t_1}\mathbf{v}_{1,t_1}$;
- $\mathbf{v}_2 := \alpha_{2,1}\mathbf{v}_{2,1}+\cdots+\alpha_{2,t_2}\mathbf{v}_{2,t_2}$;

  $\vdots$
- $\mathbf{v}_k := \alpha_{k,1}\mathbf{v}_{k,1}+\cdots+\alpha_{k,t_k}\mathbf{v}_{k,t_k}$.

So, $\mathbf{v}_1+\mathbf{v}_2+\cdots+\mathbf{v}_k=\mathbf{0}$. Now, note that for each $i\in\{1,\dots,k\}$, the vector $\mathbf{v}_i$ is a linear combination of vectors in $E_{\lambda_i}(f)$; since $E_{\lambda_i}(f)$ is a subsapce of V, it follows that $\mathbf{v}_i\in E_{\lambda_i}(f)$.

Proof. Fix $\alpha_{1,1}, \dots, \alpha_{1,t_1}, \alpha_{2,1}, \dots, \alpha_{2,t_2}, \dots, \alpha_{k,1}, \dots, \alpha_{k,t_k} \in \mathbb{F}$ s.t.

$$\sum_{i=1}^{k} \left(\alpha_{i,1}\mathbf{v}_{i,1} + \dots + \alpha_{i,t_i}\mathbf{v}_{i,t_i} \right) = \mathbf{0}.$$

$\forall i \in \{1, \dots, k\}$: set $\mathbf{v}_i := \alpha_{i,1}\mathbf{v}_{i,1} + \dots + \alpha_{i,t_i}\mathbf{v}_{i,t_i}$, that is

- $\mathbf{v}_1 := \alpha_{1,1}\mathbf{v}_{1,1} + \dots + \alpha_{1,t_1}\mathbf{v}_{1,t_1}$;
- $\mathbf{v}_2 := \alpha_{2,1}\mathbf{v}_{2,1} + \dots + \alpha_{2,t_2}\mathbf{v}_{2,t_2}$;

  $\vdots$
- $\mathbf{v}_k := \alpha_{k,1}\mathbf{v}_{k,1} + \dots + \alpha_{k,t_k}\mathbf{v}_{k,t_k}$.

So, $\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_k = \mathbf{0}$. Now, note that for each $i \in \{1, \dots, k\}$, the vector $\mathbf{v}_i$ is a linear combination of vectors in $E_{\lambda_i}(f)$; since $E_{\lambda_i}(f)$ is a subsapce of V, it follows that $\mathbf{v}_i \in E_{\lambda_i}(f)$. Consequently, $\forall i \in \{1, \dots, k\}$: $\mathbf{v}_i$ is either $\mathbf{0}$ or an eigenvector of f associated with the eigenvalue λ_i. WTS $\mathbf{v}_1 = \mathbf{v}_2 = \dots = \mathbf{v}_k = \mathbf{0}$.

Proof. Fix $\alpha_{1,1}, \dots, \alpha_{1,t_1}, \alpha_{2,1}, \dots, \alpha_{2,t_2}, \dots, \alpha_{k,1}, \dots, \alpha_{k,t_k} \in \mathbb{F}$ s.t.

$$\sum_{i=1}^{k} \left(\alpha_{i,1}\mathbf{v}_{i,1} + \dots + \alpha_{i,t_i}\mathbf{v}_{i,t_i} \right) \quad = \quad \mathbf{0}.$$

$\forall i \in \{1, \dots, k\}$: set $\mathbf{v}_i := \alpha_{i,1}\mathbf{v}_{i,1} + \dots + \alpha_{i,t_i}\mathbf{v}_{i,t_i}$, that is

- $\mathbf{v}_1 := \alpha_{1,1}\mathbf{v}_{1,1} + \dots + \alpha_{1,t_1}\mathbf{v}_{1,t_1}$;
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  $\vdots$

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So, $\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_k = \mathbf{0}$. Now, note that for each $i \in \{1, \dots, k\}$, the vector $\mathbf{v}_i$ is a linear combination of vectors in $E_{\lambda_i}(f)$; since $E_{\lambda_i}(f)$ is a subsapce of V, it follows that $\mathbf{v}_i \in E_{\lambda_i}(f)$. Consequently, $\forall i \in \{1, \dots, k\}$: $\mathbf{v}_i$ is either $\mathbf{0}$ or an eigenvector of f associated with the eigenvalue λ_i. WTS $\mathbf{v}_1 = \mathbf{v}_2 = \dots = \mathbf{v}_k = \mathbf{0}$. Suppose otherwise.

Proof. Fix $\alpha_{1,1}, \dots, \alpha_{1,t_1}, \alpha_{2,1}, \dots, \alpha_{2,t_2}, \dots, \alpha_{k,1}, \dots, \alpha_{k,t_k} \in \mathbb{F}$ s.t.

$$\sum_{i=1}^{k} \Big(\alpha_{i,1}\mathbf{v}_{i,1} + \dots + \alpha_{i,t_i}\mathbf{v}_{i,t_i} \Big) \quad = \quad \mathbf{0}.$$

$\forall i \in \{1, \dots, k\}$: set $\mathbf{v}_i := \alpha_{i,1}\mathbf{v}_{i,1} + \dots + \alpha_{i,t_i}\mathbf{v}_{i,t_i}$, that is

- $\mathbf{v}_1 := \alpha_{1,1}\mathbf{v}_{1,1} + \dots + \alpha_{1,t_1}\mathbf{v}_{1,t_1}$;
- $\mathbf{v}_2 := \alpha_{2,1}\mathbf{v}_{2,1} + \dots + \alpha_{2,t_2}\mathbf{v}_{2,t_2}$;

  $\vdots$
- $\mathbf{v}_k := \alpha_{k,1}\mathbf{v}_{k,1} + \dots + \alpha_{k,t_k}\mathbf{v}_{k,t_k}$.

So, $\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_k = \mathbf{0}$. Now, note that for each $i \in \{1, \dots, k\}$, the vector $\mathbf{v}_i$ is a linear combination of vectors in $E_{\lambda_i}(f)$; since $E_{\lambda_i}(f)$ is a subsapce of V, it follows that $\mathbf{v}_i \in E_{\lambda_i}(f)$. Consequently, $\forall i \in \{1, \dots, k\}$: $\mathbf{v}_i$ is either $\mathbf{0}$ or an eigenvector of f associated with the eigenvalue λ_i. WTS $\mathbf{v}_1 = \mathbf{v}_2 = \dots = \mathbf{v}_k = \mathbf{0}$. Suppose otherwise. By symmetry, WMA $\exists \ell \in \{1, \dots, k\}$ s.t. $\mathbf{v}_1, \dots, \mathbf{v}_\ell$ are all non-zero (and are consequently eigenvectors of f associated with $\lambda_1, \dots, \lambda_\ell$), while $\mathbf{v}_{\ell+1}, \dots, \mathbf{v}_k$ are all zero.

Proof (continued). So,

$$\mathbf{v}_1 + \cdots + \mathbf{v}_\ell = \mathbf{0},$$

and it follows that $\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\}$ is a linearly dependent set. But this contradicts Proposition 8.4.1.

Proof (continued). So,

$$\mathbf{v}_1 + \cdots + \mathbf{v}_\ell \quad = \quad \mathbf{0},$$

and it follows that $\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\}$ is a linearly dependent set. But this contradicts Proposition 8.4.1.

We have now shown that $\mathbf{v}_1 = \cdots = \mathbf{v}_k = \mathbf{0}$.

Proof (continued). So,

$$\mathbf{v}_1 + \cdots + \mathbf{v}_\ell \quad = \quad \mathbf{0},$$

and it follows that $\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\}$ is a linearly dependent set. But this contradicts Proposition 8.4.1.

We have now shown that $\mathbf{v}_1 = \cdots = \mathbf{v}_k = \mathbf{0}$. So, for all indices $i \in \{1, \ldots, k\}$, we have that

$$\alpha_{i,1}\mathbf{v}_{i,1} + \cdots + \alpha_{i,t_i}\mathbf{v}_{i,t_i} \quad = \quad \mathbf{0};$$

Proof (continued). So,

$$\mathbf{v}_1 + \cdots + \mathbf{v}_\ell \quad = \quad \mathbf{0},$$

and it follows that $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ is a linearly dependent set. But this contradicts Proposition 8.4.1.

We have now shown that $\mathbf{v}_1 = \cdots = \mathbf{v}_k = \mathbf{0}$. So, for all indices $i \in \{1, \dots, k\}$, we have that

$$\alpha_{i,1}\mathbf{v}_{i,1} + \cdots + \alpha_{i,t_i}\mathbf{v}_{i,t_i} \quad = \quad \mathbf{0};$$

since vectors $\mathbf{v}_{i,1}, \dots, \mathbf{v}_{i,t_i}$ are linearly independent, it follows that $\alpha_{i,1} = \cdots = \alpha_{i,t_i} = 0$.

Proof (continued). So,

$$\mathbf{v}_1 + \cdots + \mathbf{v}_\ell \;=\; \mathbf{0},$$

and it follows that $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ is a linearly dependent set. But this contradicts Proposition 8.4.1.

We have now shown that $\mathbf{v}_1 = \cdots = \mathbf{v}_k = \mathbf{0}$. So, for all indices $i \in \{1, \dots, k\}$, we have that

$$\alpha_{i,1}\mathbf{v}_{i,1} + \cdots + \alpha_{i,t_i}\mathbf{v}_{i,t_i} \;=\; \mathbf{0};$$

since vectors $\mathbf{v}_{i,1}, \dots, \mathbf{v}_{i,t_i}$ are linearly independent, it follows that $\alpha_{i,1} = \cdots = \alpha_{i,t_i} = 0$.

Since this holds for all indices $i \in \{1, \dots, k\}$, we deduce that the eigenvectors

$$\mathbf{v}_{1,1}, \dots, \mathbf{v}_{1,t_1}, \mathbf{v}_{2,1}, \dots, \mathbf{v}_{2,t_2}, \dots, \mathbf{v}_{k,1}, \dots, \mathbf{v}_{k,t_k}$$

are linearly independent, which is what we needed to show. □

Proposition 8.4.1

Let V be a vector space over a field $\mathbb{F}$, let $f : V \to V$ be a linear function, let $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ be pairwise distinct eigenvalues of f, associated with eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_k$, respectively. Then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent set.

Proposition 8.4.2

Let V be a vector space over a field $\mathbb{F}$, let $f : V \to V$ be a linear function, and let $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{F}$ be pairwise distinct eigenvalues of f. For each $i \in \{1, \dots, k\}$, let $\mathbf{v}_{i,1}, \dots, \mathbf{v}_{i,t_i}$ be linearly independent eigenvectors of f associated with the eigenvalue λ_i. Then the eigenvectors

$$\mathbf{v}_{1,1}, \dots, \mathbf{v}_{1,t_1}, \mathbf{v}_{2,1}, \dots, \mathbf{v}_{2,t_2}, \dots, \mathbf{v}_{k,1}, \dots, \mathbf{v}_{k,t_k}$$

are linearly independent.

Theorem 8.4.3

Let V be a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Let $f : V \to V$ be a linear function, let $\lambda_1, \dots, \lambda_k$ be all (distinct) the eigenvalues of f, and let $\mathcal{B}_1, \dots, \mathcal{B}_k$ be bases of the associated eigenspaces $E_{\lambda_1}(f), \dots, E_{\lambda_k}(f)$, respectively. Set $\mathcal{B} := \mathcal{B}_1 \cup \dots \cup \mathcal{B}_k$. Then all the following hold:

- (a) $\mathcal{B}$ is a linearly independent set of eigenvectors of f;
- (b) $\dim\big(E_{\lambda_1}(f)\big) + \dots + \dim\big(E_{\lambda_k}(f)\big) \leq n$, i.e. the sum of geometric multiplicities of the eigenvalues of f is at most n;
- (c) V has an eigenbasis associated with f iff the sum of geometric multiplicities of the eigenvalues of f is n, and in this case, $\mathcal{B}$ is such an eigenbasis;
- (d) V has an eigenbasis associated with f iff the sum of algebraic multiplicities of the eigenvalues of f is n, and the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity; in this case, $\mathcal{B}$ is an eigenbasis of V associated with the linear function f.

Theorem 8.4.3

Let V be a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Let $f : V \to V$ be a linear function, let $\lambda_1, \dots, \lambda_k$ be all (distinct) the eigenvalues of f, and let $\mathcal{B}_1, \dots, \mathcal{B}_k$ be bases of the associated eigenspaces $E_{\lambda_1}(f), \dots, E_{\lambda_k}(f)$, respectively. Set $\mathcal{B} := \mathcal{B}_1 \cup \dots \cup \mathcal{B}_k$. Then all the following hold:

- (a) $\mathcal{B}$ is a linearly independent set of eigenvectors of f;
- (b) $\dim\big(E_{\lambda_1}(f)\big) + \dots + \dim\big(E_{\lambda_k}(f)\big) \leq n$, i.e. the sum of geometric multiplicities of the eigenvalues of f is at most n;

Proof. Part (a) follows immediately from Proposition 8.4.2.

Theorem 8.4.3

Let V be a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Let $f : V \to V$ be a linear function, let $\lambda_1, \dots, \lambda_k$ be all (distinct) the eigenvalues of f, and let $\mathcal{B}_1, \dots, \mathcal{B}_k$ be bases of the associated eigenspaces $E_{\lambda_1}(f), \dots, E_{\lambda_k}(f)$, respectively. Set $\mathcal{B} := \mathcal{B}_1 \cup \dots \cup \mathcal{B}_k$. Then all the following hold:

- (a) $\mathcal{B}$ is a linearly independent set of eigenvectors of f;
- (b) $\dim\big(E_{\lambda_1}(f)\big) + \dots + \dim\big(E_{\lambda_k}(f)\big) \leq n$, i.e. the sum of geometric multiplicities of the eigenvalues of f is at most n;

Proof. Part (a) follows immediately from Proposition 8.4.2.

Part (b) follows from (a) and from the fact that, by Theorem 3.2.17(a), any linearly independent set of vectors in an n-dimensional vector space contains at most n vectors.

Theorem 8.4.3

(c) V has an eigenbasis associated with f iff the sum of geometric multiplicities of the eigenvalues of f is n, and in this case, $\mathcal{B}$ is such an eigenbasis;

Proof (continued). Let us prove (c).

Theorem 8.4.3

(c) V has an eigenbasis associated with f iff the sum of geometric multiplicities of the eigenvalues of f is n, and in this case, $\mathcal{B}$ is such an eigenbasis;

Proof (continued). Let us prove (c). Suppose first that the sum of geometric multiplicities of the eigenvalues of f is equal to n.

Theorem 8.4.3

(c) V has an eigenbasis associated with f iff the sum of geometric multiplicities of the eigenvalues of f is n, and in this case, $\mathcal{B}$ is such an eigenbasis;

Proof (continued). Let us prove (c). Suppose first that the sum of geometric multiplicities of the eigenvalues of f is equal to n. Then $\mathcal{B}$ is a linearly independent set of size n in the n-dimensional vector space V.

Theorem 8.4.3

(c) V has an eigenbasis associated with f iff the sum of geometric multiplicities of the eigenvalues of f is n, and in this case, $\mathcal{B}$ is such an eigenbasis;

Proof (continued). Let us prove (c). Suppose first that the sum of geometric multiplicities of the eigenvalues of f is equal to n. Then $\mathcal{B}$ is a linearly independent set of size n in the n-dimensional vector space V. So, by Corollary 3.2.20(a), $\mathcal{B}$ is a basis of V.

Theorem 8.4.3

(c) V has an eigenbasis associated with f iff the sum of geometric multiplicities of the eigenvalues of f is n, and in this case, $\mathcal{B}$ is such an eigenbasis;

Proof (continued). Let us prove (c). Suppose first that the sum of geometric multiplicities of the eigenvalues of f is equal to n. Then $\mathcal{B}$ is a linearly independent set of size n in the n-dimensional vector space V. So, by Corollary 3.2.20(a), $\mathcal{B}$ is a basis of V. Since all vectors in $\mathcal{B}$ are eigenvectors of f, it follows that $\mathcal{B}$ is an eigenbasis of V associated with f.

Theorem 8.4.3

- (c) V has an eigenbasis associated with f iff the sum of geometric multiplicities of the eigenvalues of f is n, and in this case, $\mathcal{B}$ is such an eigenbasis;

Proof (continued). Suppose, conversely, that V has an eigenbasis $\mathcal{C}$ associated with f;

Theorem 8.4.3

- (c) V has an eigenbasis associated with f iff the sum of geometric multiplicities of the eigenvalues of f is n, and in this case, $\mathcal{B}$ is such an eigenbasis;

Proof (continued). Suppose, conversely, that V has an eigenbasis $\mathcal{C}$ associated with f; since $\dim(V) = n$, we see that $|\mathcal{C}| = n$.

Theorem 8.4.3

- (c) V has an eigenbasis associated with f iff the sum of geometric multiplicities of the eigenvalues of f is n, and in this case, $\mathcal{B}$ is such an eigenbasis;

Proof (continued). Suppose, conversely, that V has an eigenbasis $\mathcal{C}$ associated with f; since $\dim(V) = n$, we see that $|\mathcal{C}| = n$. Since all vectors in $\mathcal{C}$ are eigenvecors of f, we see that they all belong to $E_{\lambda_1}(f) \cup \cdots \cup E_{\lambda_k}(f)$. But since the basis $\mathcal{C}$ of V is, in particular, linearly independent, we see that it cannot contain more than $\dim\Big(E_{\lambda_i}(f)\Big)$ many vectors from $E_{\lambda_i}(f)$ for any index $i \in \{1, \dots, k\}$.

Theorem 8.4.3

(c) V has an eigenbasis associated with f iff the sum of geometric multiplicities of the eigenvalues of f is n, and in this case, $\mathcal{B}$ is such an eigenbasis;

Proof (continued). Suppose, conversely, that V has an eigenbasis $\mathcal{C}$ associated with f; since $\dim(V) = n$, we see that $|\mathcal{C}| = n$. Since all vectors in $\mathcal{C}$ are eigenvecors of f, we see that they all belong to $E_{\lambda_1}(f) \cup \dots \cup E_{\lambda_k}(f)$. But since the basis $\mathcal{C}$ of V is, in particular, linearly independent, we see that it cannot contain more than $\dim\Big(E_{\lambda_i}(f)\Big)$ many vectors from $E_{\lambda_i}(f)$ for any index $i \in \{1, \dots, k\}$. So, $|\mathcal{C}| \leq \dim(E_{\lambda_1}(f)) + \dots + \dim(E_{\lambda_k}(f))$.

Theorem 8.4.3

(c) V has an eigenbasis associated with f iff the sum of geometric multiplicities of the eigenvalues of f is n, and in this case, $\mathcal{B}$ is such an eigenbasis;

Proof (continued). Suppose, conversely, that V has an eigenbasis $\mathcal{C}$ associated with f; since $\dim(V) = n$, we see that $|\mathcal{C}| = n$. Since all vectors in $\mathcal{C}$ are eigenvecors of f, we see that they all belong to $E_{\lambda_1}(f) \cup \cdots \cup E_{\lambda_k}(f)$. But since the basis $\mathcal{C}$ of V is, in particular, linearly independent, we see that it cannot contain more than $\dim\Big(E_{\lambda_i}(f)\Big)$ many vectors from $E_{\lambda_i}(f)$ for any index $i \in \{1, \dots, k\}$. So, $|\mathcal{C}| \leq \dim(E_{\lambda_1}(f)) + \cdots + \dim(E_{\lambda_k}(f))$. But now we have that

$$n \;=\; |\mathcal{C}| \;\leq\; \dim(E_{\lambda_1}(f)) + \cdots + \dim(E_{\lambda_k}(f)) \;\overset{\text{(b)}}{\leq}\; n,$$

Theorem 8.4.3

(c) V has an eigenbasis associated with f iff the sum of geometric multiplicities of the eigenvalues of f is n, and in this case, $\mathcal{B}$ is such an eigenbasis;

Proof (continued). Suppose, conversely, that V has an eigenbasis $\mathcal{C}$ associated with f; since $\dim(V) = n$, we see that $|\mathcal{C}| = n$. Since all vectors in $\mathcal{C}$ are eigenvecors of f, we see that they all belong to $E_{\lambda_1}(f) \cup \cdots \cup E_{\lambda_k}(f)$. But since the basis $\mathcal{C}$ of V is, in particular, linearly independent, we see that it cannot contain more than $\dim\Big(E_{\lambda_i}(f)\Big)$ many vectors from $E_{\lambda_i}(f)$ for any index $i \in \{1, \dots, k\}$. So, $|\mathcal{C}| \leq \dim(E_{\lambda_1}(f)) + \cdots + \dim(E_{\lambda_k}(f))$. But now we have that

$$n = |\mathcal{C}| \leq \dim(E_{\lambda_1}(f)) + \cdots + \dim(E_{\lambda_k}(f)) \overset{\text{(b)}}{\leq} n,$$

and it follows that $\dim(E_{\lambda_1}(f)) + \cdots + \dim(E_{\lambda_k}(f)) = n$,

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- (c) V has an eigenbasis associated with f iff the sum of geometric multiplicities of the eigenvalues of f is n, and in this case, $\mathcal{B}$ is such an eigenbasis;

Proof (continued). Suppose, conversely, that V has an eigenbasis $\mathcal{C}$ associated with f; since $\dim(V) = n$, we see that $|\mathcal{C}| = n$. Since all vectors in $\mathcal{C}$ are eigenvecors of f, we see that they all belong to $E_{\lambda_1}(f) \cup \cdots \cup E_{\lambda_k}(f)$. But since the basis $\mathcal{C}$ of V is, in particular, linearly independent, we see that it cannot contain more than $\dim\Big(E_{\lambda_i}(f)\Big)$ many vectors from $E_{\lambda_i}(f)$ for any index $i \in \{1, \dots, k\}$. So, $|\mathcal{C}| \leq \dim(E_{\lambda_1}(f)) + \cdots + \dim(E_{\lambda_k}(f))$. But now we have that

$$n \;=\; |\mathcal{C}| \;\leq\; \dim(E_{\lambda_1}(f)) + \cdots + \dim(E_{\lambda_k}(f)) \;\overset{\text{(b)}}{\leq}\; n,$$

and it follows that $\dim(E_{\lambda_1}(f)) + \cdots + \dim(E_{\lambda_k}(f)) = n$, i.e. the sum of geometric multiplicities of the eigenvalues of f is n. This proves (c).

Theorem 8.4.3

(d) V has an eigenbasis associated with f iff the sum of algebraic multiplicities of the eigenvalues of f is n, and the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity; in this case, $\mathcal{B}$ is an eigenbasis of V associated with the linear function f.

Proof (continued). It remains to prove (d). If the sum of algebraic multiplicities of the eigenvalues of f is equal to n, and the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity, then obviously, the sum of geometric multiplicities of f is equal to n,

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Proof (continued). It remains to prove (d). If the sum of algebraic multiplicities of the eigenvalues of f is equal to n, and the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity, then obviously, the sum of geometric multiplicities of f is equal to n, and so by (c), V has an eigenbasis associated with f, and $\mathcal{B}$ is one such eigenbasis.

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For the converse, assume that V has an eigenbasis $\mathcal{C}$ associated with f.

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For the converse, assume that V has an eigenbasis $\mathcal{C}$ associated with f. Let $\lambda_1, \dots, \lambda_k$ be the eigenvalues of f, with geometric multiplicities $g_1, \dots, g_k$, respectively, and algebraic multiplicities $a_1, \dots, a_k$, respectively.

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(d) V has an eigenbasis associated with f iff the sum of algebraic multiplicities of the eigenvalues of f is n, and the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity; in this case, $\mathcal{B}$ is an eigenbasis of V associated with the linear function f.

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Proof (continued). By (c), we have that $g_1 + \cdots + g_k = n$. On the other hand, the characteristic polynomial of f is of degree n, we see that the sum of algebraic multiplicitis of f is at most n, i.e. $a_1 + \cdots + a_k \leq n$.

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Proof (continued). By (c), we have that $g_1 + \cdots + g_k = n$. On the other hand, the characteristic polynomial of f is of degree n, we see that the sum of algebraic multiplicitis of f is at most n, i.e. $a_1 + \cdots + a_k \leq n$. But by Theorem 8.2.17, the geometric multiplicity of an eigenvalue of f is no greater than the algebraic multiplicity of that eigenvalue, that is, $g_i \leq a_i$ for all indices $i \in \{1, \dots, n\}$.

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$$n = g_1 + \cdots + g_k \leq a_1 + \cdots + a_k \leq n,$$

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$$n \;=\; g_1 + \cdots + g_k \;\leq\; a_1 + \cdots + a_k \;\leq\; n,$$

and we deduce that $a_1 + \cdots + a_k = n$ and that $g_i = a_i$ for all $i \in \{1, \dots, k\}$. This proves (d). $\square$

Theorem 8.4.3

Let V be a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. Let $f : V \to V$ be a linear function, let $\lambda_1, \dots, \lambda_k$ be all (distinct) the eigenvalues of f, and let $\mathcal{B}_1, \dots, \mathcal{B}_k$ be bases of the associated eigenspaces $E_{\lambda_1}(f), \dots, E_{\lambda_k}(f)$, respectively. Set $\mathcal{B} := \mathcal{B}_1 \cup \dots \cup \mathcal{B}_k$. Then all the following hold:

- (a) $\mathcal{B}$ is a linearly independent set of eigenvectors of f;
- (b) $\dim\big(E_{\lambda_1}(f)\big) + \dots + \dim\big(E_{\lambda_k}(f)\big) \leq n$, i.e. the sum of geometric multiplicities of the eigenvalues of f is at most n;
- (c) V has an eigenbasis associated with f iff the sum of geometric multiplicities of the eigenvalues of f is n, and in this case, $\mathcal{B}$ is such an eigenbasis;
- (d) V has an eigenbasis associated with f iff the sum of algebraic multiplicities of the eigenvalues of f is n, and the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity; in this case, $\mathcal{B}$ is an eigenbasis of V associated with the linear function f.

Corollary 8.4.4

Let V be a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. If a linear function $f : V \to V$ has n distinct eigenvalues, then V has an eigenbasis associated with f.

Proof.

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Proof. Let $f : V \to V$ be a linear function that has n distinct eigenvalues, say $\lambda_1, \dots, \lambda_n$.

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By the definition of an eigenvalue, we have that $\dim\big(E_{\lambda_i}(f)\big) \geq 1$ for all $i \in \{1, \ldots, n\}$.

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By the definition of an eigenvalue, we have that $\dim\big(E_{\lambda_i}(f)\big) \geq 1$ for all $i \in \{1, \ldots, n\}$. Consequently, $\dim\big(E_{\lambda_1}(f)\big) + \cdots + \dim\big(E_{\lambda_n}(f)\big) \geq n$.

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On the other hand, Theorem 8.4.3(b) guarantees that $\dim\big(E_{\lambda_1}(f)\big) + \cdots + \dim\big(E_{\lambda_n}(f)\big) \leq n$.

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Proof. Let $f : V \to V$ be a linear function that has n distinct eigenvalues, say $\lambda_1, \dots, \lambda_n$.

By the definition of an eigenvalue, we have that $\dim(E_{\lambda_i}(f)) \geq 1$ for all $i \in \{1, \dots, n\}$. Consequently,
$\dim(E_{\lambda_1}(f)) + \cdots + \dim(E_{\lambda_n}(f)) \geq n$.

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Corollary 8.4.4

Let V be a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and set $n := \dim(V)$. If a linear function $f : V \to V$ has n distinct eigenvalues, then V has an eigenbasis associated with f.

Proof. Let $f : V \to V$ be a linear function that has n distinct eigenvalues, say $\lambda_1, \dots, \lambda_n$.

By the definition of an eigenvalue, we have that $\dim(E_{\lambda_i}(f)) \geq 1$ for all $i \in \{1, \dots, n\}$. Consequently, $\dim(E_{\lambda_1}(f)) + \dots + \dim(E_{\lambda_n}(f)) \geq n$.

On the other hand, Theorem 8.4.3(b) guarantees that $\dim(E_{\lambda_1}(f)) + \dots + \dim(E_{\lambda_n}(f)) \leq n$.

Thus, $\dim(E_{\lambda_1}(f)) + \dots + \dim(E_{\lambda_n}(f)) = n$, and so by Theorem 8.4.3(c), V has an eigenbasis associated with f. □

- We would now like to "translate" Theorem 8.4.3 and Corollary 8.4.4 into the language of matrices.

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- Given a field $\mathbb{F}$ and a square matrix $A \in \mathbb{F}^{n \times n}$, we can define $f_A : \mathbb{F}^n \to \mathbb{F}^n$ by setting $f_A(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{F}^n$.
    - So, f_A is linear, and its standard matrix is A.

- We would now like to "translate" Theorem 8.4.3 and Corollary 8.4.4 into the language of matrices.
- Given a field $\mathbb{F}$ and a square matrix $A \in \mathbb{F}^{n \times n}$, we can define $f_A : \mathbb{F}^n \to \mathbb{F}^n$ by setting $f_A(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{F}^n$.
    - So, f_A is linear, and its standard matrix is A.
- We can apply Theorem 8.4.3 and Corollary 8.4.4 to the linear function f_A, and then get the same result for A "for free."
    - Next two slides!

Theorem 8.4.5

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$. Let $\lambda_1, \dots, \lambda_k$ be all the (distinct) eigenvalues of A, and let $\mathcal{B}_1, \dots, \mathcal{B}_k$ be bases of the associated eigenspaces $E_{\lambda_1}(A), \dots, E_{\lambda_k}(A)$, respectively. Set $\mathcal{B} := \mathcal{B}_1 \cup \dots \cup \mathcal{B}_k$. Then all the following hold:

a. $\mathcal{B}$ is a linearly independent set of eigenvectors of A;
b. $\dim\big(E_{\lambda_1}(A)\big) + \dots + \dim\big(E_{\lambda_k}(A)\big) \leq n$, i.e. the sum of geometric multiplicities of the eigenvalues of A is at most n;
c. $\mathbb{F}^n$ has an eigenbasis associated with A iff the sum of geometric multiplicities of the eigenvalues of A is n, and in this case, $\mathcal{B}$ is such an eigenbasis;
d. $\mathbb{F}^n$ has an eigenbasis associated with A iff the sum of algebraic multiplicities of the eigenvalues of A is n, and the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity; in this case, $\mathcal{B}$ is an eigenbasis of $\mathbb{F}^n$ associated with the matrix A.

Corollary 8.4.6

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. If A has n distinct eigenvalues, then $\mathbb{F}^n$ has an eigenbasis associated with A.

3. Diagonalization

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Definition

For a field $\mathbb{F}$, a square matrix $D \in \mathbb{F}^{n\times n}$ is *diagonal* if all its entries off the main diagonal are zero (the entries on the main diagonal may or may not be zero). For scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{F}$, $D(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the $n \times n$ matrix with $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the main diagonal (appearing in that order) and 0's everywhere else, i.e.

$$\begin{aligned} D(\lambda_1, \lambda_2, \ldots, \lambda_n) &:= \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1\mathbf{e}_1 & \ldots & \lambda_n\mathbf{e}_n \end{bmatrix}, \end{aligned}$$

where as usual, $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the standard basis vectors of $\mathbb{F}^n$.

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- Note that diagonal matrices are, in particular, triangular.

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$$\begin{aligned} D(\lambda_1, \lambda_2, \ldots, \lambda_n) &:= \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 \mathbf{e}_1 & \ldots & \lambda_n \mathbf{e}_n \end{bmatrix}, \end{aligned}$$

where as usual, $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the standard basis vectors of $\mathbb{F}^n$.

- Note that diagonal matrices are, in particular, triangular.
- So, Propositions 7.3.1 and 8.2.7 (next slide) apply.

Proposition 7.3.1

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ be a triangular matrix in $\mathbb{F}^{n \times n}$. Then $\det(A) = \prod_{i=1}^{n} a_{i,i} = a_{1,1}a_{2,2} \dots a_{n,n}$, that is, $\det(A)$ is equal to the product of entries on the main diagonal of A.

Proposition 7.3.1

Let $\mathbb{F}$ be a field, and let $A = \left[\begin{array}{c} a_{i,j} \end{array} \right]_{n \times n}$ be a triangular matrix in $\mathbb{F}^{n \times n}$. Then $\det(A) = \prod_{i=1}^{n} a_{i,i} = a_{1,1}a_{2,2} \dots a_{n,n}$, that is, $\det(A)$ is equal to the product of entries on the main diagonal of A.

Proposition 8.2.7

Let $\mathbb{F}$ be a field, and let $A = \left[\begin{array}{c} a_{i,j} \end{array} \right]_{n \times n}$ be a triangular matrix in $\mathbb{F}^{n \times n}$. Then the characteristic polynomial of A is

$$p_A(\lambda) = \prod_{i=1}^{n} (\lambda - a_{i,i}) = (\lambda - a_{1,1})(\lambda - a_{2,2}) \dots (\lambda - a_{n,n}),$$

the eigenvalues of A are precisely the entries of A on its main diagonal, and moreover, the algebraic multiplicity of each eigenvalue is precisely the number of times that it appears on the main diagonal of A. Consequently, the spectrum of A is $\{a_{1,1}, a_{2,2}, \dots, a_{n,n}\}$, i.e. the multiset formed precisely by the main diagonal entries of A, with each number appearing in the spectrum of A the same number of times as on the main diagonal of A.

- Thus, for scalars $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ (where $\mathbb{F}$ is a field), and for the diagonal matrix $D := D(\lambda_1, \dots, \lambda_n)$, we have the following:
  - $\det(D) = \lambda_1 \dots \lambda_n$;
  - $p_D(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n)$.

- Thus, for scalars $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ (where $\mathbb{F}$ is a field), and for the diagonal matrix $D := D(\lambda_1, \dots, \lambda_n)$, we have the following:
  - $\det(D) = \lambda_1 \dots \lambda_n$;
  - $p_D(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n)$.

- We now state three simple propositions about diagonal matrices.
  - The proofs are easy and we omit them here.
  - However, the proofs can be found in the Lecture Notes.

Proposition 8.5.1

Let $\mathbb{F}$ be a field, let $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ ($n \geq 1$) be arbitrary scalars, and set $D := D(\lambda_1, \dots, \lambda_n)$. Then both the following hold:

- (a) for all vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ in $\mathbb{F}^n$, we have that

$$D\mathbf{x} \;=\; \begin{bmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{bmatrix};$$

- (b) for all matrices $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$ in $\mathbb{F}^{m \times n}$, we have that

$$AD \;=\; \begin{bmatrix} \lambda_1 \mathbf{a}_1 & \dots & \lambda_n \mathbf{a}_n \end{bmatrix}.$$

- Proof: Lecture Notes (easy!).

Proposition 8.5.2

Let $\mathbb{F}$ be a field, and let $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n \in \mathbb{F}$ ($n \geq 1$) be arbitrary scalars. Then

$$D(\lambda_1, \ldots, \lambda_n)\, D(\mu_1, \ldots, \mu_n) \quad = \quad D(\lambda_1\mu_1, \ldots, \lambda_n\mu_n).$$

- Proof: Lecture Notes (easy!)

Proposition 8.5.2

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$$D(\lambda_1, \dots, \lambda_n)\, D(\mu_1, \dots, \mu_n) \;=\; D(\lambda_1\mu_1, \dots, \lambda_n\mu_n).$$

- Proof: Lecture Notes (easy!)

Proposition 8.5.3

Let $\mathbb{F}$ be a field, let $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ $(n \geq 1)$, and set $D := D(\lambda_1, \dots, \lambda_n)$. Then both the following hold:

(a) for all non-negative integers m, we have that $D^m = D(\lambda_1^m, \dots, \lambda_n^m)$;

(b) D is invertible iff $\lambda_1, \dots, \lambda_n$ are all non-zero, and in this case, we have that $D^m = D(\lambda_1^m, \dots, \lambda_n^m)$ for all integers m.

- Proof: Lecture Notes (easy!)

Theorem 8.5.4

Let V be a non-trivial, finite-dimensional vector space, let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of V, and let $f : V \to V$ be a linear function. Then $\mathcal{B}$ is an eigenbasis of V associated with f iff the matrix ${}_{\mathcal{B}}[\ f\ ]_{\mathcal{B}}$ is diagonal. Moreover, in this case, we have that

$$ {}_{\mathcal{B}}[\ f\ ]_{\mathcal{B}} \ = \ D(\lambda_1, \dots, \lambda_n), $$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of f associated with the eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, respectively.

Proof.

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where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of f associated with the eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, respectively.

Proof. Suppose first that $\mathcal{B}$ is an eigenbasis of V associated with f.

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$$ {}_{\mathcal{B}}[\ f\ ]_{\mathcal{B}} = D(\lambda_1, \dots, \lambda_n), $$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of f associated with the eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, respectively.

Proof. Suppose first that $\mathcal{B}$ is an eigenbasis of V associated with f. Then, by definition, vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are eigenvectors of f, and we let $\lambda_1, \dots, \lambda_n$, respectively, be the associated eigenvalues.

Theorem 8.5.4

Let V be a non-trivial, finite-dimensional vector space, let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of V, and let $f : V \to V$ be a linear function. Then $\mathcal{B}$ is an eigenbasis of V associated with f iff the matrix ${}_{\mathcal{B}}[\ f\ ]_{\mathcal{B}}$ is diagonal. Moreover, in this case, we have that

$${}_{\mathcal{B}}[\ f\ ]_{\mathcal{B}} \;=\; D(\lambda_1, \dots, \lambda_n),$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of f associated with the eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, respectively.

Proof. Suppose first that $\mathcal{B}$ is an eigenbasis of V associated with f. Then, by definition, vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are eigenvectors of f, and we let $\lambda_1, \dots, \lambda_n$, respectively, be the associated eigenvalues. Then $f(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$ for all indices $i \in \{1, \dots, n\}$, and we have the following (next slide):

Theorem 8.5.4

Let V be a non-trivial, finite-dimensional vector space, let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of V, and let $f : V \to V$ be a linear function. Then $\mathcal{B}$ is an eigenbasis of V associated with f iff the matrix ${}_{\mathcal{B}}[\ f\ ]_{\mathcal{B}}$ is diagonal. Moreover, in this case, we have that

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where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of f associated with the eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, respectively.

Proof (continued).

$$\begin{aligned} {}_{\mathcal{B}}[\ f\ ]_{\mathcal{B}} &= \left[\ [\ f(\mathbf{v}_1)\ ]_{\mathcal{B}} \ \dots \ [\ f(\mathbf{v}_1)\ ]_{\mathcal{B}}\ \right] \qquad \text{by Theorem 4.5.1} \\ &= \left[\ [\ \lambda_1\mathbf{v}_1\ ]_{\mathcal{B}} \ \dots \ [\ \lambda_n\mathbf{v}_n\ ]_{\mathcal{B}}\ \right] \\ &= \left[\ \lambda_1\mathbf{e}_1 \ \dots \ \lambda_n\mathbf{e}_n\ \right] \\ &= D(\lambda_1, \dots, \lambda_n). \end{aligned}$$

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Proof (continued). Conversely, suppose that the matrix ${}_{\mathcal{B}}[\ f\ ]_{\mathcal{B}}$ is diagonal, and let $\lambda_1, \dots, \lambda_n$ be the entries of this matrix on the main diagonal, so that

$$ {}_{\mathcal{B}}[\ f\ ]_{\mathcal{B}} \quad = \quad D(\lambda_1, \dots, \lambda_n) \quad = \quad \begin{bmatrix} \lambda_1\mathbf{e}_1 & \dots & \lambda_n\mathbf{e}_n \end{bmatrix}. $$

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We will show that the basis vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are eigenvectors of f with associated eigenvalues $\lambda_1, \dots, \lambda_n$, respectively.

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We will show that the basis vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are eigenvectors of f with associated eigenvalues $\lambda_1, \dots, \lambda_n$, respectively. Fix any index $i \in \{1, \dots, n\}$; WTS $f(\mathbf{v}_i) = \lambda_i\mathbf{v}_i$. Since $\mathbf{v}_i$ is the i-th basis vector of $\mathcal{B}$, we have that $\big[\ \mathbf{v}_i\ \big]_{\mathcal{B}} = \mathbf{e}_i$.

Theorem 8.5.4

Let V be a non-trivial, finite-dimensional vector space, let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of V, and let $f : V \to V$ be a linear function. Then $\mathcal{B}$ is an eigenbasis of V associated with f iff the matrix ${}_{\mathcal{B}}[\ f\ ]_{\mathcal{B}}$ is diagonal. Moreover, in this case, we have that

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We will show that the basis vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are eigenvectors of f with associated eigenvalues $\lambda_1, \dots, \lambda_n$, respectively. Fix any index $i \in \{1, \dots, n\}$; WTS $f(\mathbf{v}_i) = \lambda_i\mathbf{v}_i$. Since $\mathbf{v}_i$ is the i-th basis vector of $\mathcal{B}$, we have that $[\ \mathbf{v}_i\ ]_{\mathcal{B}} = \mathbf{e}_i$. We now compute (next slide):

Theorem 8.5.4

Let V be a non-trivial, finite-dimensional vector space, let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of V, and let $f : V \to V$ be a linear function. Then $\mathcal{B}$ is an eigenbasis of V associated with f iff the matrix ${}_{\mathcal{B}}\big[\ f\ \big]_{\mathcal{B}}$ is diagonal. Moreover, in this case, we have that

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where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of f associated with the eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, respectively.

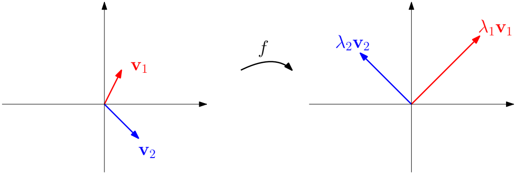
Proof (continued).

$$ \begin{aligned} \big[\ f(\mathbf{v}_i)\ \big]_{\mathcal{B}} \ &= \ {}_{\mathcal{B}}\big[\ f\ \big]_{\mathcal{B}}\big[\ \mathbf{v}_i\ \big]_{\mathcal{B}} \ = \ \big[\ \lambda_1\mathbf{e}_1 \ \ \dots \ \ \lambda_n\mathbf{e}_n\ \big]\,\mathbf{e}_i \\ &\overset{(*)}{=} \ \lambda_i\mathbf{e}_i \ = \ \lambda_i\big[\ \mathbf{v}_i\ \big]_{\mathcal{B}} \ \overset{(**)}{=} \ \big[\ \lambda_i\mathbf{v}_i\ \big]_{\mathcal{B}}, \end{aligned} $$

where (*) follows from Proposition 1.4.4, and (**) follows from the linearity of $\big[\ \cdot\ \big]_{\mathcal{B}}$. Since $\big[\ \cdot\ \big]_{\mathcal{B}}$ is an isomorphism (and in particular, one-to-one), it follows that $f(\mathbf{v}_i) = \lambda_i\mathbf{v}_i$, which is what we needed to show. □

- **Remark:** Suppose that V is a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$.

- **Remark:** Suppose that V is a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$.
  - By Theorems 4.3.2 and 8.5.4, linear functions from V to V that have a diagonal matrix are precisely those that can be defined starting from some basis, and then scaling each of the basis elements.



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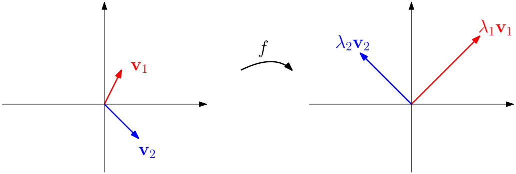

  - Indeed, suppose that $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is some basis of V, and that $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ are some scalars.

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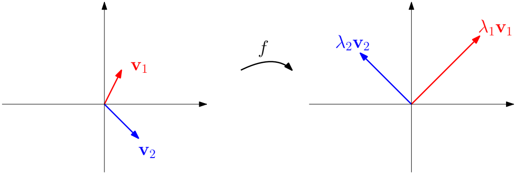

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- **Remark:** Suppose that V is a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$.
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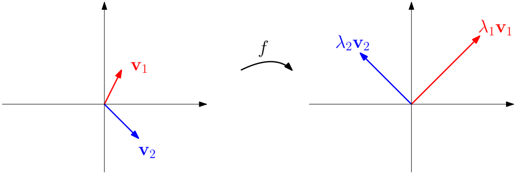

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    - But then by Theorem 8.5.4, ${}_{\mathcal{B}}\big[\ f\ \big]_{\mathcal{B}} = D(\lambda_1, \ldots, \lambda_n)$.

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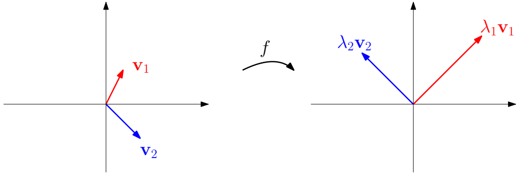

  - Indeed, suppose that $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is some basis of V, and that $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ are some scalars.
  - By Theorem 4.3.2, there exists a unique linear function $f : V \to V$ such that $f(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$.
  - But then by Theorem 8.5.4, ${}_{\mathcal{B}}[\ f\ ]_{\mathcal{B}} = D(\lambda_1, \ldots, \lambda_n)$.
  - By Theorem 8.5.4, the converse also holds.

Definition

A matrix $A \in \mathbb{F}^{n \times n}$ (where $\mathbb{F}$ is a field) is *diagonalizable* if it is similar to a diagonal matrix. To *diagonalize* a diagonalizable matrix A means to compute a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$ (equivalently: $A = PDP^{-1}$).

Theorem 8.5.6

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$ be a matrix. Then A is diagonalizable if and only if $\mathbb{F}^n$ has an eigenbasis associated with A. Moreover, if $\mathcal{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ is any eigenbasis of $\mathbb{F}^n$ associated with A, and $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A associated with the eigenvectors $\mathbf{p}_1, \dots, \mathbf{p}_n$, respectively, then

$$D = P^{-1}AP \qquad \text{and} \qquad A = PDP^{-1},$$

where $D = D(\lambda_1, \dots, \lambda_n)$ and $P = \begin{bmatrix} \mathbf{p}_1 & \dots & \mathbf{p}_n \end{bmatrix}$.

- Proof: Lecture Notes.

Theorem 8.5.6

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$ be a matrix. Then A is diagonalizable if and only if $\mathbb{F}^n$ has an eigenbasis associated with A. Moreover, if $\mathcal{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ is any eigenbasis of $\mathbb{F}^n$ associated with A, and $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A associated with the eigenvectors $\mathbf{p}_1, \dots, \mathbf{p}_n$, respectively, then

$$D = P^{-1}AP \qquad \text{and} \qquad A = PDP^{-1},$$

where $D = D(\lambda_1, \dots, \lambda_n)$ and $P = \begin{bmatrix} \mathbf{p}_1 & \dots & \mathbf{p}_n \end{bmatrix}$.

- Proof: Lecture Notes.
  - Theorem 8.5.6 can be obtained as a corollary of Theorem 8.5.4 (try it!).
  - However, in the Lecture Notes, there is a proof "from scratch" (i.e. one that uses matrices only).

Theorem 8.5.6

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$ be a matrix. Then A is diagonalizable if and only if $\mathbb{F}^n$ has an eigenbasis associated with A. Moreover, if $\mathcal{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ is any eigenbasis of $\mathbb{F}^n$ associated with A, and $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A associated with the eigenvectors $\mathbf{p}_1, \dots, \mathbf{p}_n$, respectively, then

$$D \quad = \quad P^{-1}AP \qquad \text{and} \qquad A \quad = \quad PDP^{-1},$$

where $D = D(\lambda_1, \dots, \lambda_n)$ and $P = \begin{bmatrix} \mathbf{p}_1 & \dots & \mathbf{p}_n \end{bmatrix}$.

Corollary 8.5.7

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$. If A has n distinct eigenvalues, then A is diagonalizable.

Proof.

Theorem 8.5.6

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$ be a matrix. Then A is diagonalizable if and only if $\mathbb{F}^n$ has an eigenbasis associated with A. Moreover, if $\mathcal{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ is any eigenbasis of $\mathbb{F}^n$ associated with A, and $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A associated with the eigenvectors $\mathbf{p}_1, \dots, \mathbf{p}_n$, respectively, then

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Corollary 8.5.7

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. If A has n distinct eigenvalues, then A is diagonalizable.

Proof. Assume that A has n distinct eigenvalues. By Corollary 8.4.6, $\mathbb{F}^n$ has an eigenbasis associated with A. So, by Theorem 8.5.6, A is diagonalizable. □

- Theorems 8.4.5 and 8.5.6 together give us a recipe for determining whether a matrix $A \in \mathbb{F}^{n \times n}$ is diagonalizable, and if so, for diagonalizing it (i.e. for finding a diagonal matrix D and an invertible matrix P, both in $\mathbb{F}^{n \times n}$, such that $D = P^{-1}AP$).
- We proceed as follows (next two slides).

1. We compute the characteristic polynomial $p_A(\lambda)$ and its roots. By Theorem 8.2.2, the roots of $p_A(\lambda)$ are the eigenvalues of A, and we can read off the algebraic multiplicities of those eigenvalues from the polynomial $p_A(\lambda)$.
    - Computing the roots of $p_A(\lambda)$ is the computationally tricky part, since there is no formula for computing the roots of a high-degree polynomial. If we cannot figure out how to compute the roots of $p_A(\lambda)$, then we are stuck: the matrix A may or may not be diagonalizable, but computationally, we cannot diagonalize it.

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2. If the sum of algebraic multiplicities of the eigenvalues of A is less than n, then by Theorem 8.4.5, $\mathbb{F}^n$ does not have an eigenbasis associated with A, and so by Theorem 8.5.6, A is not diagonalizable.

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2. If the sum of algebraic multiplicities of the eigenvalues of A is less than n, then by Theorem 8.4.5, $\mathbb{F}^n$ does not have an eigenbasis associated with A, and so by Theorem 8.5.6, A is not diagonalizable.
3. From now on, we assume that the sum of algebraic multiplicities of the eigenvalues of A, call them $\lambda_1, \dots, \lambda_k$, is n. We then compute a basis $\mathcal{B}_i$ for each eigenspace $E_{\lambda_i}(A)$, which allows us to compute the geometric multiplicities of all the eigenvalues of A.

4. If the geometric multiplicity of some eigenvalue of A is smaller than its algebraic multiplicity, then by Theorem 8.4.5, $\mathbb{F}^n$ does not have an eigenbasis associated with A, and so by Theorem 8.5.6, A is not diagonalizable.

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5. From now on, we assume that the geometric multiplicity of each eigenvalue of A is equal to its algebraic multiplicity. Theorem 8.4.5 then guarantees that $\mathbb{F}^n$ has an eigenbasis associated with A, and moreover, that $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ is one such eigenbasis.

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6. By Theorem 8.5.6, A is diagonalizable. We now follow the recipe from Theorem 8.5.6 to actually diagonalize A.
7. We form the matrix P whose columns are precisely the vectors in the eigenbasis $\mathcal{B}$. We form the diagonal matrix D, where on the main diagonal we place the eigenvalues of A, taking care that, for each $i \in \{1, \dots, n\}$, the i-th entry on the main diagonal of D is the eigenvalue associated with the i-th column of P (which is, by construction, an eigenvector of A). Now $D = P^{-1}AP$.

Example 8.5.8.

Consider the following matrix in $\mathbb{C}^{3\times 3}$:

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}.$$

Determine whether A is diagonalizable, and if so, diagonalize it.

Solution.

Example 8.5.8.

Consider the following matrix in $\mathbb{C}^{3\times 3}$:

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}.$$

Determine whether A is diagonalizable, and if so, diagonalize it.

Solution. The matrix A is precisely the matrix from Example 8.2.4. In that example, we determined that A has two eigenvalues:

- $\lambda_1 = 4$ (with algebraic multiplicity 1 and geometric multiplicity 1);
- $\lambda_2 = 5$ (with algebraic multiplicity 2 and geometric multiplicity 2).

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- $\lambda_1 = 4$ (with algebraic multiplicity 1 and geometric multiplicity 1);
- $\lambda_2 = 5$ (with algebraic multiplicity 2 and geometric multiplicity 2).

Since the sum of algebraic multiplicities of the eigenvalues of A is 3, and since the geometric multiplicity of each eigenvalue of A is equal to its algebraic multiplicity, we see that the 3×3 matrix A is indeed diagonalizable.

Solution (continued). Reminder: $\lambda_1 = 4$, $\lambda_2 = 5$.

In Example 8.2.4, we saw that:

- $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ is a basis of the eigespace $E_{\lambda_1}(A)$;
- $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of the eigenspace $E_{\lambda_2}(A)$.

Solution (continued). Reminder: $\lambda_1 = 4$, $\lambda_2 = 5$.

In Example 8.2.4, we saw that:

- $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ is a basis of the eigespace $E_{\lambda_1}(A)$;
- $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of the eigenspace $E_{\lambda_2}(A)$.

So, we set

$$D \;:=\; \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad P \;:=\; \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and we see that $D = P^{-1}AP$. $\square$

Example 8.5.9

Consider the following matrix in $\mathbb{C}^{5\times 5}$:

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Determine whether A is diagonalizable, and if so, diagonalize it.

Solution.

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Solution. The matrix A is precisely the matrix from Example 8.2.8.

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Determine whether A is diagonalizable, and if so, diagonalize it.

Solution. The matrix A is precisely the matrix from Example 8.2.8. In that example, we determined that A has three eigenvalues:

- $\lambda_1 = 1$ (with alg. mult. 2 and geom. mult. 2);
- $\lambda_2 = 2$ (with alg. mult. 1 and geom. mult. 1);
- $\lambda_3 = 3$ (with alg. mult. 2 and geom. mult. 1).

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Since the geometric multiplicity of the eigenvalue $\lambda_3 = 3$ is strictly smaller than the algebraic multiplicity, we see that A is not diagonalizable. □

- Suppose that we have successfully diagonalized a square matrix $A \in \mathbb{F}^{n \times n}$ (where $\mathbb{F}$ is a field), that is, that we have computed a diagonal matrix D and an invertible matrix P, both in $\mathbb{F}^{n \times n}$, such that $D = P^{-1}AP$.

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- Then we can easily read off the spectrum and a basis of each eigenspace of A, as Proposition 8.5.12 (next slide) shows.

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- This proposition essentially summarizes various facts about diagonalizable matrices that we have proven already, but it is convenient to have them stated in one proposition.

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- This proposition essentially summarizes various facts about diagonalizable matrices that we have proven already, but it is convenient to have them stated in one proposition.
  - The proof is in the Lecture Notes. Here, we omit it.

Proposition 8.5.12

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Assume that $D = P^{-1}AP$, where $D = D(\lambda_1, \dots, \lambda_n)$ is a diagonal and $P = \begin{bmatrix} \mathbf{p}_1 & \dots & \mathbf{p}_n \end{bmatrix}$ an invertible matrix, both in $\mathbb{F}^{n \times n}$. Then the characteristic polynomial of A is

$$p_A(\lambda) \;=\; \prod_{i=1}^{n} (\lambda - \lambda_i) \;=\; (\lambda - \lambda_1) \dots (\lambda - \lambda_n),$$

and the spectrum of A is $\{\lambda_1, \dots, \lambda_n\}$. Moreover, for each eigenvalue λ_0 of A,[a] the algebraic and geometric multiplicity of λ_0 are both equal to the number of times that λ_0 appears on the main diagonal of D, and moreover, if λ_0 appears precisely in positions $i_1, \dots, i_k$ of the main diagonal of D, then the corresponding columns of P (i.e. vectors $\mathbf{p}_{i_1}, \dots, \mathbf{p}_{i_k}$) form a basis of the eigenspace $E_{\lambda_0}(A)$. Finally, $\{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ is an eigenbasis of $\mathbb{F}^n$ associated with the matrix A.

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[a] So, $\lambda_0 \in \{\lambda_1, \dots, \lambda_n\}$, since $\{\lambda_1, \dots, \lambda_n\}$ is the spectrum of A.

Example 8.5.13

Consider the following matrices in $\mathbb{C}^{6\times 6}$ (color coded for emphasis):

$$D = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 3 & 8 & 8 & 3 & 4 \\ 2 & 8 & 0 & 0 & 0 & 2 \\ 5 & 4 & 6 & 4 & 5 & 0 \\ 0 & 5 & 8 & 5 & 4 & 3 \\ 1 & 0 & 8 & 0 & 3 & 0 \\ 0 & 2 & 0 & 3 & 0 & 2 \end{bmatrix}.$$

It can be checked that P is invertible (for example, we can compute that $\det(P) = -1020 \neq 0$, and so by Theorem 7.4.1, P is invertible). We now set $A = PDP^{-1}$, so that $D = P^{-1}AP$. Then by Proposition 8.5.12, all the following hold (next three slides):

$$D = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 3 & 8 & 8 & 3 & 4 \\ 2 & 8 & 0 & 0 & 0 & 2 \\ 5 & 4 & 6 & 4 & 5 & 0 \\ 0 & 5 & 8 & 5 & 4 & 3 \\ 1 & 0 & 8 & 0 & 3 & 0 \\ 0 & 2 & 0 & 3 & 0 & 2 \end{bmatrix}.$$

Example 8.5.13 (continued)

- the characteristic polynomial of A is

$$p_A(\lambda) = (\lambda - 3)(\lambda - 4)^3(\lambda - 5)^2;$$

- the spectrum of A is $\{5, 4, 5, 3, 4, 4\}$, which we can optionally reorder as $\{3, 4, 4, 4, 5, 5\}$;
- the eigenvalues of A are 3 (with algebraic and geometric multiplicity 1), 4 (with algebraic and geometric multiplicity 3), and 5 (with algebraic and geometric multiplicity 2);

$$D = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 3 & 8 & 8 & 3 & 4 \\ 2 & 8 & 0 & 0 & 0 & 2 \\ 5 & 4 & 6 & 4 & 5 & 0 \\ 0 & 5 & 8 & 5 & 4 & 3 \\ 1 & 0 & 8 & 0 & 3 & 0 \\ 0 & 2 & 0 & 3 & 0 & 2 \end{bmatrix}.$$

Example 8.5.13 (continued)

- we can read off bases of the eigenspaces $E_3(A)$, $E_4(A)$, and $E_5(A)$, as follows:
  - a basis of $E_3(A)$ is $\left\{ \begin{bmatrix} 8 \\ 0 \\ 4 \\ 5 \\ 0 \\ 3 \end{bmatrix} \right\}$,
  - a basis of $E_4(A)$ is $\left\{ \begin{bmatrix} 3 \\ 8 \\ 4 \\ 5 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \\ 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$,
  - a basis of $E_5(A)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 0 \\ 6 \\ 8 \\ 8 \\ 0 \end{bmatrix} \right\}$;

$$D = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 3 & 8 & 8 & 3 & 4 \\ 2 & 8 & 0 & 0 & 0 & 2 \\ 5 & 4 & 6 & 4 & 5 & 0 \\ 0 & 5 & 8 & 5 & 4 & 3 \\ 1 & 0 & 8 & 0 & 3 & 0 \\ 0 & 2 & 0 & 3 & 0 & 2 \end{bmatrix}.$$

Example 8.5.13 (continued)

- the columns of P form an eigenbasis of $\mathbb{C}^n$ associated with the matrix A.