Linear Algebra 2

Lecture #19

Laplace expansion. Cramer's rule. The adjugate matrix

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- This lecture has four parts:

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    1. The multiplicative property of determinants

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  2. Laplace expansion

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$$\det(AB) = \det(A)\det(B).$$

- To prove Theorem 7.5.2, we first need a technical proposition (next slide).
- Recall that an *elementary matrix* is any matrix obtained by performing one elementary row operation on an identity matrix I_n.
  - Here, it is possible that $E = I_n$. In this case, we can take R to be the multiplication of the first row by the scalar 1.

Proposition 7.5.1

Let $\mathbb{F}$ be a field, let $A, E \in \mathbb{F}^{n \times n}$, and assume that E is an elementary matrix. Then $\det(EA) = \det(E)\det(A)$.

Proof.

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By Proposition 1.11.11(a), EA is the matrix obtained by performing R on A.

Now, by Theorem 7.3.2, there exists some scalar $\alpha \in \mathbb{F} \setminus \{0\}$ s.t. for any matrix $M \in \mathbb{F}^{n \times n}$, if M_R is the matrix obtained by performing R on M, then $\det(M_R) = \alpha \det(M)$.

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- $\det(E) = \alpha \det(I_n) = \alpha$;
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- $\det(E) = \alpha\det(I_n) = \alpha$;
- $\det(EA) = \alpha\det(A)$.

It follows that

$$\det(EA) \quad = \quad \alpha\det(A) \quad = \quad \det(E)\det(A).$$

□

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Proof. Suppose first that at least one of A, B is non-invertible. Then by Corollary 3.3.16, AB is also non-invertible.

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Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n\times n}$. Then

$$\det(AB) \;=\; \det(A)\det(B).$$

Proof. Suppose first that at least one of A, B is non-invertible. Then by Corollary 3.3.16, AB is also non-invertible. But by Theorem 7.4.1, non-invertible matrices have determinant zero, and so $\det(AB) = 0 = \det(A)\det(B)$.

- If A is non-invertible, then $\det(A) = 0$.
- If B is non-invertible, then $\det(B) = 0$.
- In either case, $\det(A)\det(B) = 0$.

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Proof (continued). From now on, we assume that A and B are both invertible.

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$$\det(AB) \;=\; \det(A)\det(B).$$

Proof (continued). From now on, we assume that A and B are both invertible. Therefore, by the Invertible Matrix Theorem, each of them can be written as a product of elementary matrices, say $A = E_1^A \dots E_p^A$ and $B = E_1^B \dots E_q^B$, where $E_1^A, \dots, E_p^A, E_1^B, \dots, E_q^B$ are elementary matrices.

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- $\det(A) = \det(E_1^A) \dots \det(E_p^A)$;
- $\det(B) = \det(E_1^B) \dots \det(E_q^B)$;
- $\det(AB) = \det(E_1^A) \dots \det(E_p^A)\det(E_1^B) \dots \det(E_q^B)$.

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Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n\times n}$. Then

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Proof (continued). From now on, we assume that A and B are both invertible. Therefore, by the Invertible Matrix Theorem, each of them can be written as a product of elementary matrices, say $A = E_1^A \dots E_p^A$ and $B = E_1^B \dots E_q^B$, where $E_1^A, \dots, E_p^A, E_1^B, \dots, E_q^B$ are elementary matrices. So, $AB = E_1^A \dots E_p^A E_1^B \dots E_q^B$. By repeatedly applying Proposition 7.5.1, we get that

- $\det(A) = \det(E_1^A) \dots \det(E_p^A)$;
- $\det(B) = \det(E_1^B) \dots \det(E_q^B)$;
- $\det(AB) = \det(E_1^A) \dots \det(E_p^A)\det(E_1^B) \dots \det(E_q^B)$.

But now

$$\det(AB) \;=\; \underbrace{\det(E_1^A)\dots\det(E_p^A)}_{=\det(A)}\underbrace{\det(E_1^B)\dots\det(E_q^B)}_{=\det(B)} \;=\; \det(A)\det(B),$$

which is what we needed to show. □

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Corollary 7.5.3

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$ be an invertible matrix. Then

$$\det(A^{-1}) \;=\; \frac{1}{\det(A)}.$$

Proof.

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Proof. Since $AA^{-1} = I_n$, we see that

$$\det(A)\det(A^{-1}) \overset{\text{Thm. 7.5.2}}{=} \det(AA^{-1}) \;=\; \det(I_n) \;=\; 1.$$

We now see that $\det(A^{-1}) = \frac{1}{\det(A)}$, which is what we needed to show. $\square$

- Reminder:

Definition

Matrices $A, B \in \mathbb{F}^{n \times n}$ (where $\mathbb{F}$ is a field) are said to be *similar* if there exists an invertible matrix $P \in \mathbb{F}^{n \times n}$ s.t. $B = P^{-1}AP$.

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Corollary 7.5.4

Let $\mathbb{F}$ be a field, and let A and B be similar matrices in $\mathbb{F}^{n \times n}$. Then $\det(A) = \det(B)$.

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Let $\mathbb{F}$ be a field, and let A and B be similar matrices in $\mathbb{F}^{n \times n}$. Then $\det(A) = \det(B)$.

Proof. Since A and B are similar, there exists an invertible matrix $P \in \mathbb{F}^{n \times n}$ s.t. $B = P^{-1}AP$. We then have that

$$\begin{aligned} \det(B) &= \det(P^{-1}AP) \\ &= \det(P^{-1})\det(A)\det(P) && \text{by Theorem 7.5.2} \\ &= \frac{1}{\det(P)}\det(A)\det(P) && \text{by Corollary 7.5.3} \\ &= \det(A). \end{aligned}$$

□

- Reminder:

Theorem 4.5.16

Let $\mathbb{F}$ be a field, let $B, C \in \mathbb{F}^{n\times n}$ be matrices, and let V be an n-dimensional vector space over the field $\mathbb{F}$. Then the following are equivalent:

(a) B and C are similar;

(b) for all bases $\mathcal{B}$ of V and linear functions $f : V \to V$ s.t. $B = {}_{\mathcal{B}}[\ f\ ]_{\mathcal{B}}$, there exists a basis $\mathcal{C}$ of V s.t. $C = {}_{\mathcal{C}}[\ f\ ]_{\mathcal{C}}$;

(c) for all bases $\mathcal{C}$ of V and linear functions $f : V \to V$ s.t. $C = {}_{\mathcal{C}}[\ f\ ]_{\mathcal{C}}$, there exists a basis $\mathcal{B}$ of V s.t. $B = {}_{\mathcal{B}}[\ f\ ]_{\mathcal{B}}$;

(d) there exist bases $\mathcal{B}$ and $\mathcal{C}$ of V and a linear function $f : V \to V$ s.t. $B = {}_{\mathcal{B}}[\ f\ ]_{\mathcal{B}}$ and $C = {}_{\mathcal{C}}[\ f\ ]_{\mathcal{C}}$.

Definition

Suppose that V is a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and that $f : V \to V$ is a linear function. Then we define the determinant of f to be

$$\det(f) \;\; := \;\; \det\Big({}_{\mathcal{B}}\big[\ f\ \big]_{\mathcal{B}} \Big),$$

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- Let us explain why this is well-defined, that is, why the value of $\det(f)$ that we get depends only on f, and not on the particular choice of the basis $\mathcal{B}$.

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- Suppose that $\mathcal{C}$ is any basis of V.
- Then by Theorem 4.5.16, matrices ${}_{\mathcal{B}}\big[\ f\ \big]_{\mathcal{B}}$ and ${}_{\mathcal{C}}\big[\ f\ \big]_{\mathcal{C}}$ are similar, and consequently (by Corollary 7.5.4), they have the same determinant.

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- So, $\det(f)$ is well-defined.

Definition

Suppose that V is a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and that $f : V \to V$ is a linear function. Then we define the determinant of f to be

$$\det(f) \;\; := \;\; \det\Big({}_{\mathcal{B}}\big[\; f \;\big]_{\mathcal{B}} \Big),$$

where $\mathcal{B}$ is any basis of V.

- **Remark:** Note that we defined determinants only for linear functions whose domain and codomain are one and the same, and moreover, are finite-dimensional and non-null.

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Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n \times n}$. Then

$$\det(AB) = \det(A)\det(B).$$

Corollary 7.5.5

Let A be an orthogonal matrix in $\mathbb{R}^{n \times n}$. Then $\det(A) = \pm 1$ (i.e. $\det(A)$ is either $+1$ or -1).

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$$1 \;=\; \det(I_n) \;=\; \det(A^T A) \;\overset{(*)}{=}\; \det(A^T)\det(A) \;\overset{(**)}{=}\; \det(A)^2,$$

where (*) follows from Theorem 7.5.2, and (**) follows from Theorem 7.1.3.

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where (*) follows from Theorem 7.5.2, and (**) follows from Theorem 7.1.3. But now we see that $\det(A) = \pm 1$, which is what we needed to show. □

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Let A be an orthogonal matrix in $\mathbb{R}^{n\times n}$. Then $\det(A) = \pm 1$ (i.e. $\det(A)$ is either $+1$ or -1).

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  - For example, the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

satisfies $\det(A) = 1$, but A is not orthogonal.

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  - Then by Theorem 7.4.1, we have that $\det(A) \neq 0$.

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  - Then by Theorem 7.4.1, we have that $\det(A) \neq 0$.
  - We now form the matrix B by multiplying one row or one column of A by $\frac{1}{\det(A)}$, and we see that $\det(B) = 1$.

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Let A be an orthogonal matrix in $\mathbb{R}^{n \times n}$. Then $\det(A) = \pm 1$ (i.e. $\det(A)$ is either $+1$ or -1).

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  - For example, the matrix

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    satisfies $\det(A) = 1$, but A is not orthogonal.
  - More generally, suppose that A is **any** invertible matrix in $\mathbb{R}^{n \times n}$.
  - Then by Theorem 7.4.1, we have that $\det(A) \neq 0$.
  - We now form the matrix B by multiplying one row or one column of A by $\frac{1}{\det(A)}$, and we see that $\det(B) = 1$.
  - However, B need not be orthogonal.

2. Laplace expansion

Definition

For a matrix $A = \left[\ a_{i,j}\ \right]_{n\times n}$ (where $n \geq 2$) with entries in some field $\mathbb{F}$, and for indices $p, q \in \{1, \ldots, n\}$, $A_{p,q}$ is the $(n-1)\times(n-1)$ matrix obtained from A by deleting the p-th row and q-th column.

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,q-1} & a_{1,q} & a_{1,q+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{p-1,1} & \cdots & a_{p-1,q-1} & a_{p-1,q} & a_{p-1,q+1} & \cdots & a_{p-1,n} \\ a_{p,1} & \cdots & a_{p,q-1} & a_{p,q} & a_{p,q+1} & \cdots & a_{p,n} \\ a_{p+1,1} & \cdots & a_{p+1,q-1} & a_{p+1,q} & a_{p+1,q+1} & \cdots & a_{p+1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,q-1} & a_{n,q} & a_{n,q+1} & \cdots & a_{n,n} \end{bmatrix}$$

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- **Terminology:** The determinants

$$\det(A_{i,j}), \qquad \text{with } i, j \in \{1, \ldots, n\}$$

are referred to as the *first minors* of A, whereas numbers

$$C_{i,j} \quad := \quad (-1)^{i+j} \det(A_{i,j}) \qquad \text{with } i, j \in \{1, \ldots, n\}$$

are referred to as the *cofactors* of A.

Laplace expansion

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n \times n}$. Then both the following hold:

(a) **[expansion along the i-th row]** for all $i \in \{1, \dots, n\}$:

$$\det(A) \quad = \quad \sum_{j=1}^{n} (-1)^{i+j}\ a_{i,j}\ \det(A_{i,j});$$

(b) **[expansion along the j-th column]** for all $j \in \{1, \dots, n\}$:

$$\det(A) \quad = \quad \sum_{i=1}^{n} (-1)^{i+j}\ a_{i,j}\ \det(A_{i,j}).$$

- **Remark:** If we write $C_{i,j} := (-1)^{i+j}\det(A_{i,j})$ for all $i, j \in \{1, \dots, n\}$ (so, the $C_{i,j}$'s are the cofactors of A), then the formula from (a) becomes $\det(A) = \sum_{j=1}^{n} a_{i,j} C_{i,j}$, and the formula from (b) becomes $\det(A) = \sum_{i=1}^{n} a_{i,j} C_{i,j}$.
- This is why Laplace expansion is also referred to as "cofactor expansion."

Laplace expansion

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n \times n}$. Then both the following hold:

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$$\det(A) \quad = \quad \sum_{i=1}^{n} (-1)^{i+j}\ a_{i,j}\ \det(A_{i,j}).$$

- First an example, then a proof.

Example 7.6.3

Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \\ 7 & 0 & 8 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Compute $\det(A)$ in two ways:

(a) via Laplace expansion along the third row;

(b) via Laplace expansion along the second column.

Solution. (a) Laplace expansion along the third row:

$$\begin{aligned}\det(A) &= \begin{vmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \\ 7 & 0 & 8 \end{vmatrix} \\ &= (-1)^{3+1}\,7\begin{vmatrix} 0 & 1 \\ 4 & 5 \end{vmatrix} + (-1)^{3+2}\,0\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} + (-1)^{3+3}\,8\begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix} \\ &= 7\underbrace{\begin{vmatrix} 0 & 1 \\ 4 & 5 \end{vmatrix}}_{=-4} + 8\underbrace{\begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix}}_{=8} = 36.\end{aligned}$$

Solution (continued). (b) Laplace expansion along the second column:

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \\ 7 & 0 & 8 \end{vmatrix} \\
&= (-1)^{1+2}\, 0 \begin{vmatrix} 3 & 5 \\ 7 & 8 \end{vmatrix} + (-1)^{2+2}\, 4 \begin{vmatrix} 2 & 1 \\ 7 & 8 \end{vmatrix} + (-1)^{3+2}\, 0 \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} \\
&= 4 \underbrace{\begin{vmatrix} 2 & 1 \\ 7 & 8 \end{vmatrix}}_{=9} = 36.
\end{aligned}$$

□

Example 7.6.3

Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \\ 7 & 0 & 8 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Compute $\det(A)$ in two ways:

(a) via Laplace expansion along the third row;

(b) via Laplace expansion along the second column.

- As a general rule, it is best to expand along a row or column that has a lot of zeros (if such a row or column exists), since that reduces the amount of calculation that we need to perform.
  - So, in Example 7.6.3, it was easier to expand along the second column.

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with entries understood to be in $\mathbb{R}$. Compute $\det(A)$ in two ways:

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- As a general rule, it is best to expand along a row or column that has a lot of zeros (if such a row or column exists), since that reduces the amount of calculation that we need to perform.
  - So, in Example 7.6.3, it was easier to expand along the second column.
- See the Lecture Notes for another example (with a larger matrix).

Laplace expansion

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j} \ \right]_{n \times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n \times n}$. Then both the following hold:

(a) **[expansion along the *i*-th row]** for all $i \in \{1, \dots, n\}$:

$$\det(A) \quad = \quad \sum_{j=1}^{n} (-1)^{i+j} \; a_{i,j} \; \det(A_{i,j});$$

(b) **[expansion along the *j*-th column]** for all $j \in \{1, \dots, n\}$:

$$\det(A) \quad = \quad \sum_{i=1}^{n} (-1)^{i+j} \; a_{i,j} \; \det(A_{i,j}).$$

- Let's prove this!
- We begin with a technical proposition.

Proposition 7.6.1

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{(n-1)\times(n-1)}$ (where $n \geq 2$) and $\mathbf{a} \in \mathbb{F}^{n-1}$. Then

$$\det\left(\left[\begin{array}{c:c} A & \mathbf{0} \\ \hdashline \mathbf{a}^T & 1 \end{array} \right]_{n\times n} \right) \;=\; \det(A).$$

Proof.

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Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{(n-1)\times(n-1)}$ (where $n \geq 2$) and $\mathbf{a} \in \mathbb{F}^{n-1}$. Then

$$\det\left(\left[\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{a}^T & 1 \end{array} \right]_{n\times n} \right) = \det(A).$$

Proof. First, set $\left[\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{a}^T & 1 \end{array} \right]_{n\times n} = \left[\begin{array}{c} a_{i,j} \end{array} \right]_{n\times n}$, so that all the following hold:

- $A = \left[\begin{array}{c} a_{i,j} \end{array} \right]_{(n-1)\times(n-1)}$;
- $a_{n,n} = 1$;
- for all $i \in \{1, \dots, n-1\}$, $a_{i,n} = 0$;
- for all $j \in \{1, \dots, n-1\}$, $a_{n,j}$ is the j-th entry of the vector $\mathbf{a}$.

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Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{(n-1)\times(n-1)}$ (where $n \geq 2$) and $\mathbf{a} \in \mathbb{F}^{n-1}$. Then

$$\det\left(\left[\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{a}^T & 1 \end{array} \right]_{n\times n} \right) \;=\; \det(A).$$

Proof (continued). Next, for all $\sigma \in S_{n-1}$, let $\sigma^* \in S_n$ be given by

- $\sigma^*(i) = \sigma(i)$ for all $i \in \{1, \dots, n-1\}$,
- $\sigma^*(n) = n$.

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- $\sigma^*(i) = \sigma(i)$ for all $i \in \{1, \dots, n-1\}$,
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So, for any $\sigma \in S_{n-1}$, the disjoint cycle decomposition of σ^* is obtained by adding the one-element cycle (n) to the disjoint cycle decomposition of σ, and consequently, $\text{sgn}(\sigma) = \text{sgn}(\sigma^*)$.

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Proof (continued). Next, for all $\sigma \in S_{n-1}$, let $\sigma^* \in S_n$ be given by

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Set

$$S_n^* \;:=\; \{\sigma^* \mid \sigma \in S_{n-1}\} \;=\; \{\pi \in S_n \mid \pi(n) = n\}.$$

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Proof (continued). Next, for all $\sigma \in S_{n-1}$, let $\sigma^* \in S_n$ be given by

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Set

$$S_n^* := \{\sigma^* \mid \sigma \in S_{n-1}\} = \{\pi \in S_n \mid \pi(n) = n\}.$$

We then have the following (next slide):

Proof (continued).

$$\begin{aligned}
\det(A) &= \sum_{\sigma \in S_{n-1}} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} \cdots a_{n-1,\sigma(n-1)} \\
&= \sum_{\sigma \in S_{n-1}} \operatorname{sgn}(\sigma^*) a_{1,\sigma^*(1)} \cdots a_{n-1,\sigma^*(n-1)} \underbrace{a_{n,\sigma^*(n)}}_{=1} \\
&= \sum_{\pi \in S_n^*} \operatorname{sgn}(\pi) a_{1,\pi(1)} \cdots a_{n-1,\pi(n-1)} a_{n,\pi(n)} \\
&\overset{(*)}{=} \sum_{\pi \in S_n} \operatorname{sgn}(\pi) a_{1,\pi(1)} \cdots a_{n-1,\pi(n-1)} a_{n,\pi(n)} \\
&= \det\Bigg(\left[\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{a}^T & 1 \end{array} \right]_{n \times n} \Bigg),
\end{aligned}$$

where (*) follows from the fact that for all $\pi \in S_n \setminus S_n^*$, we have that $a_{n,\pi(n)} = 0$. □

Proposition 7.6.1

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{(n-1)\times(n-1)}$ (where $n \geq 2$) and $\mathbf{a} \in \mathbb{F}^{n-1}$. Then

$$\det\left(\left[\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{a}^T & 1 \end{array} \right]_{n\times n} \right) \;=\; \det(A).$$

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$$\det\left(\left[\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{a}^T & 1 \end{array} \right]_{n \times n} \right) = \det(A).$$

- Reminder:

Theorem 7.1.3

Let $\mathbb{F}$ be a field. For all $A \in \mathbb{F}^{n\times n}$, we have that $\det(A^T) = \det(A)$.

Laplace expansion

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j} \ \right]_{n \times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n \times n}$. Then both the following hold:

(a) **[expansion along the i-th row]** for all $i \in \{1, \dots, n\}$:

$$\det(A) \quad = \quad \sum_{j=1}^{n} (-1)^{i+j} \ a_{i,j} \ \det(A_{i,j});$$

(b) **[expansion along the j-th column]** for all $j \in \{1, \dots, n\}$:

$$\det(A) \quad = \quad \sum_{i=1}^{n} (-1)^{i+j} \ a_{i,j} \ \det(A_{i,j}).$$

Proof.

Laplace expansion

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n \times n}$. Then both the following hold:

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$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \, a_{i,j} \, \det(A_{i,j}).$$

Proof. In view of Theorem 7.1.3, it is enough to prove (b).

Laplace expansion

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n \times n}$. Then both the following hold:

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Proof. In view of Theorem 7.1.3, it is enough to prove (b).

Fix $j \in \{1, \dots, n\}$. We must show that

$$\det(A) \;=\; \sum_{i=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j}).$$

Proof (cont.). Reminder: WTS $\det(A) = \sum_{i=1}^{n} (-1)^{i+j}\ a_{i,j} \det(A_{i,j})$.

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First, set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$.

Proof (cont.). Reminder: WTS $\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \; a_{i,j} \; \det(A_{i,j})$.

First, set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$. Then $\mathbf{a}_j = \sum_{i=1}^{n} a_{i,j} \mathbf{e}_i$, *Proof (cont.).* Reminder: WTS $\det(A) = \sum_{i=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j})$.

First, set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$. Then $\mathbf{a}_j = \sum_{i=1}^{n} a_{i,j}\mathbf{e}_i$, and so

$$\begin{aligned}
\det(A) &= \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_j & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix} \right) \\
&= \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \sum_{i=1}^{n} a_{i,j}\mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix} \right) \\
&\overset{(*)}{=} \sum_{i=1}^{n} a_{i,j} \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix} \right),
\end{aligned}$$

where (*) follows from Proposition 7.2.1(a).

Proof (cont.). Reminder: WTS $\det(A) = \sum_{i=1}^{n} (-1)^{i+j}\ a_{i,j}\ \det(A_{i,j})$.

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\end{aligned}$$

where (*) follows from Proposition 7.2.1(a).

Fix an arbitrary index $i \in \{1, \dots, n\}$.

Proof (cont.). Reminder: WTS $\det(A) = \sum_{i=1}^{n} (-1)^{i+j}\ a_{i,j}\ \det(A_{i,j})$.

First, set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$. Then $\mathbf{a}_j = \sum_{i=1}^{n} a_{i,j}\mathbf{e}_i$, and so

$$\begin{aligned}
\det(A) &= \det\Big(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_j & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix}\Big) \\
&= \det\Big(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \sum_{i=1}^{n} a_{i,j}\mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix}\Big) \\
&\overset{(*)}{=} \sum_{i=1}^{n} a_{i,j}\det\Big(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix}\Big),
\end{aligned}$$

where (*) follows from Proposition 7.2.1(a).

Fix an arbitrary index $i \in \{1, \dots, n\}$. To complete the proof, it now suffices to show that

$$\det\Big(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix}\Big) = (-1)^{i+j} \det(A_{i,j}).$$

Proof (continued). Reminder: WTS

$$\det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix} \right) = (-1)^{i+j} \det(A_{i,j}).$$

Proof (continued). Reminder: WTS

$$\det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix}\right) = (-1)^{i+j} \det(A_{i,j}).$$

By iteratively performing $n - j$ column swaps on the matrix

$$B_i := \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix},$$

we can obtain the matrix

$$C_i := \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n & \mathbf{e}_i \end{bmatrix}.$$

Proof (continued). Reminder: WTS

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$$C_i := \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n & \mathbf{e}_i \end{bmatrix}.$$

By iteratively performing $n - i$ row swaps on the matrix C_i, we can obtain the matrix

$$\left[\begin{array}{c|c} A_{i,j} & \mathbf{0} \\ \hline \mathbf{a}^T & 1 \end{array} \right],$$

where $\mathbf{a}^T$ is the row vector of length $n - 1$ obtained from the i-th row of A by deleting its j-th entry.

Proof (continued). Since swapping two rows or two columns has the effect of changing the sign of the determinant, we see that

$$
\begin{aligned}
\det(B_i) &= (-1)^{n-j}\det(C_i) \\
&= (-1)^{n-j}(-1)^{n-i}\det\left(\left[\begin{array}{c:c} A_{i,j} & \mathbf{0} \\ \hdashline \mathbf{a}^T & 1 \end{array} \right] \right) \\
&\overset{(*)}{=} (-1)^{2n-i-j}\det(A_{i,j}) \\
&= (-1)^{i+j}\det(A_{i,j}),
\end{aligned}
$$

where (*) follows from Proposition 7.6.1. This completes the argument. □

Laplace expansion

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j} \ \right]_{n \times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n \times n}$. Then both the following hold:

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$$\det(A) \quad = \quad \sum_{j=1}^{n} (-1)^{i+j} \ a_{i,j} \ \det(A_{i,j});$$

(b) **[expansion along the *j*-th column]** for all $j \in \{1, \dots, n\}$:

$$\det(A) \quad = \quad \sum_{i=1}^{n} (-1)^{i+j} \ a_{i,j} \ \det(A_{i,j}).$$

Theorem 7.6.6

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$ and $B \in \mathbb{F}^{m \times m}$ be square matrices. Then

$$\det\left(\left[\begin{array}{c:c} A & O_{n \times m} \\ \hdashline O_{m \times n} & B \end{array} \right] \right) \;\; = \;\; \det(A)\det(B).$$

Proof (outline).

Theorem 7.6.6

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$ and $B \in \mathbb{F}^{m\times m}$ be square matrices. Then

$$\det\left(\left[\begin{array}{c|c} A & O_{n\times m} \\ \hline O_{m\times n} & B \end{array}\right]\right) = \det(A)\det(B).$$

Proof (outline). This can be proven (for example) by induction on n, via Laplace expansion along the leftmost column. The details are left as an exercise. □

Theorem 7.6.6

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$ and $B \in \mathbb{F}^{m \times m}$ be square matrices. Then

$$\det\left(\left[\begin{array}{c|c} A & O_{n \times m} \\ \hline O_{m \times n} & B \end{array}\right]\right) = \det(A)\det(B).$$

Corollary 7.6.7

Let $\mathbb{F}$ be a field, and let $A_1 \in \mathbb{F}^{n_1 \times n_1}, A_2 \in \mathbb{F}^{n_2 \times n_2}, \dots, A_k \in \mathbb{F}^{n_k \times n_k}$ be square matrices. Then

$$\det\left(\left[\begin{array}{c|c|c|c} A_1 & O_{n_1 \times n_2} & \dots & O_{n_1 \times n_k} \\ \hline O_{n_2 \times n_1} & A_2 & \dots & O_{n_2 \times n_k} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline O_{n_k \times n_1} & O_{n_k \times n_2} & \dots & A_k \end{array}\right]\right) = \prod_{i=1}^{k} \det(A_i).$$

Proof. This follows from Theorem 7.6.6 via an easy induction on k. □

- Intermission: Fraction notation in fields.

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- Let $\mathbb{F}$ be a field.

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- Let $\mathbb{F}$ be a field.
- For $a \in \mathbb{F} \setminus \{0\}$, we sometimes use the notation $\frac{1}{a}$ instead of a^{-1} (the multiplicative inverse of a in the field $\mathbb{F}$).
  - For instance, in $\mathbb{Z}_3$, we have $\frac{1}{1} = 1^{-1} = 1$ and $\frac{1}{2} = 2^{-1} = 2$ (because in $\mathbb{Z}_3$, we have that $2 \cdot 2 = 1$).

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- In a similar vein, for scalars $a, b \in \mathbb{F}$ such that $b \neq 0$, we sometimes write $\frac{a}{b}$ instead of $b^{-1}a$.
  - For example, in $\mathbb{Z}_5$, we have that $3^{-1} = 2$ (because $3 \cdot 2 = 1$), and so $\frac{4}{3} = 3^{-1} \cdot 4 = 2 \cdot 4 = 3$.

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  - For example, in $\mathbb{Z}_5$, we have that $3^{-1} = 2$ (because $3 \cdot 2 = 1$), and so $\frac{4}{3} = 3^{-1} \cdot 4 = 2 \cdot 4 = 3$.
- It is sometimes more convenient to use the notation $\frac{1}{a}$ instead of a^{-1}, and $\frac{a}{b}$ instead of $b^{-1}a$.
- However, when working over a finite field such as $\mathbb{Z}_p$ (for a prime number p), we **never** leave a fraction as a final answer, and instead, we always simplify.

3. Cramer's rule

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    - For example, for

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix},$$

we have that

$$A_1(\mathbf{b}) = \begin{bmatrix} 4 & 1 & 1 \\ 5 & 2 & 2 \\ 6 & 0 & 3 \end{bmatrix}, \qquad A_2(\mathbf{b}) = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 5 & 2 \\ 0 & 6 & 3 \end{bmatrix}, \qquad A_3(\mathbf{b}) = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 6 \end{bmatrix}.$$

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- In what follows, it will be convenient to use the fraction notation in fields.

Cramer's rule

Let $\mathbb{F}$ be a field, and let A be an invertible matrix in $\mathbb{F}^{n\times n}$, and let $\mathbf{b} \in \mathbb{F}^n$. Then the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely

$$\mathbf{x} \quad = \quad \left[\begin{array}{cccc} \frac{\det\big(A_1(\mathbf{b})\big)}{\det(A)} & \frac{\det\big(A_2(\mathbf{b})\big)}{\det(A)} & \cdots & \frac{\det\big(A_n(\mathbf{b})\big)}{\det(A)} \end{array} \right]^T .$$

- First an example, then a proof.

Example 7.7.1

Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_3$. Solve the matrix-vector equation $A\mathbf{x} = \mathbf{b}$.

Solution.

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$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_3$. Solve the matrix-vector equation $A\mathbf{x} = \mathbf{b}$.

Solution. Note that $\det(A) = 2$, and in particular, A is invertible (by Theorem 7.4.1). So, Cramer's rule applies. We compute:

- $\det\big(A_1(\mathbf{b})\big) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & 1 \end{vmatrix} = 2;$
- $\det\big(A_2(\mathbf{b})\big) = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 1;$
- $\det\big(A_3(\mathbf{b})\big) = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0.$

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with entries understood to be in $\mathbb{Z}_3$. Solve the matrix-vector equation $A\mathbf{x} = \mathbf{b}$.

Solution (continued). By Cramer's rule, $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} \frac{\det\left(A_1(\mathbf{b})\right)}{\det(A)} & \frac{\det\left(A_2(\mathbf{b})\right)}{\det(A)} & \frac{\det\left(A_3(\mathbf{b})\right)}{\det(A)} \end{bmatrix}^T \\ &= \begin{bmatrix} \frac{2}{2} & \frac{1}{2} & \frac{0}{2} \end{bmatrix}^T \\ &= \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^T. \end{aligned}$$

□

Cramer's rule

Let $\mathbb{F}$ be a field, and let A be an invertible matrix in $\mathbb{F}^{n\times n}$, and let $\mathbf{b} \in \mathbb{F}^n$. Then the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely

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Proof. Since A is invertible, we know that the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.

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Fix an index $j \in \{1, \dots, n\}$. WTS

$$x_j \quad = \quad \frac{\det\big(A_j(\mathbf{b})\big)}{\det(A)}.$$

Proof (continued). Set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$. Then:

$$\begin{aligned}
\det(A_j(\mathbf{b})) &= \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{b} & \mathbf{a}_{j+1} & \dots \mathbf{a}_n \end{bmatrix} \right) \\
&= \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & A\mathbf{x} & \mathbf{a}_{j+1} & \dots \mathbf{a}_n \end{bmatrix} \right) \\
&= \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \sum_{i=1}^{n} x_i \mathbf{a}_i & \mathbf{a}_{j+1} & \dots \mathbf{a}_n \end{bmatrix} \right) \\
&\overset{(*)}{=} \sum_{i=1}^{n} x_i \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_i & \mathbf{a}_{j+1} & \dots \mathbf{a}_n \end{bmatrix} \right) \\
&\overset{(**)}{=} x_j \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_j & \mathbf{a}_{j+1} & \dots \mathbf{a}_n \end{bmatrix} \right) \\
&= x_j \det(A),
\end{aligned}$$

where (*) follows from Proposition 7.2.1(a), and (**) follows from the fact that any matrix with two identical columns has determinant zero (by Proposition 7.1.5).

Cramer's rule

Let $\mathbb{F}$ be a field, and let A be an invertible matrix in $\mathbb{F}^{n\times n}$, and let $\mathbf{b} \in \mathbb{F}^n$. Then the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely

$$\mathbf{x} = \begin{bmatrix} \frac{\det\left(A_1(\mathbf{b})\right)}{\det(A)} & \frac{\det\left(A_2(\mathbf{b})\right)}{\det(A)} & \dots & \frac{\det\left(A_n(\mathbf{b})\right)}{\det(A)} \end{bmatrix}^T.$$

Proof (continued). We have now shown that

$$\det\big(A_j(\mathbf{b})\big) = x_j \det(A).$$

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Proof (continued). We have now shown that

$$\det\left(A_j(\mathbf{b})\right) \;=\; x_j \det(A).$$

Since A is invertible, Theorem 7.4.1 guarantees that $\det(A) \neq 0$.

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Proof (continued). We have now shown that

$$\det(A_j(\mathbf{b})) = x_j \det(A).$$

Since A is invertible, Theorem 7.4.1 guarantees that $\det(A) \neq 0$. So, we can divide both sides of the equality above by $\det(A)$ to obtain

$$x_j = \frac{\det\left(A_j(\mathbf{b})\right)}{\det(A)}.$$

This completes the argument. □

4. The adjugate matrix

Definition

Given a field $\mathbb{F}$ and a matrix $A \in \mathbb{F}^{n\times n}$ ($n \geq 2$), with cofactors $C_{i,j} = (-1)^{i+j}\det(A_{i,j})$ (for $i, j \in \{1, \dots, n\}$), the *cofactor matrix* of A is the matrix $\left[\ C_{i,j}\ \right]_{n\times n}$. The *adjugate matrix* (also called the *classical adjoint*) of A, denoted by $\mathrm{adj}(A)$, is the transpose of the cofactor matrix of A, i.e.

$$\mathrm{adj}(A) \quad := \quad \left(\left[\ C_{i,j}\ \right]_{n\times n}\right)^T.$$

So, the i, j-th entry of $\mathrm{adj}(A)$ is the cofactor $C_{j,i}$ (note the swapping of the indices).

Example 7.8.1

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Compute the cofactor and adjugate matrices of the matrix A.

Solution.

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Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Compute the cofactor and adjugate matrices of the matrix A.

Solution. For all $i, j \in \{1, 2, 3\}$, we let $C_{i,j} = (-1)^{i+j}\det(A_{i,j})$. We compute (next slide):

Solution (continued). Reminder: $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

- $C_{1,1} = (-1)^{1+1} \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = 6$;
- $C_{1,2} = (-1)^{1+2} \begin{vmatrix} 0 & 2 \\ 0 & 3 \end{vmatrix} = 0$;
- $C_{1,3} = (-1)^{1+3} \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0$;
- $C_{2,1} = (-1)^{2+1} \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = -3$;
- $C_{2,2} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = 3$;
- $C_{2,3} = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$;
- $C_{3,1} = (-1)^{3+1} \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$;
- $C_{3,2} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2$;
- $C_{3,3} = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$.

Example 7.8.1

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Compute the cofactor and adjugate matrices of the matrix A.

Solution (continued). So, the cofactor matrix of A is

$$\begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ -3 & 3 & 0 \\ 0 & -2 & 2 \end{bmatrix}.$$

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The adjugate matrix of A is the transpose of the cofactor matrix, i.e.

$$\text{adj}(A) = \begin{bmatrix} 6 & -3 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix}.$$

□

Theorem 7.8.2

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$ $(n \geq 2)$. Then

$$\text{adj}(A)\, A \;=\; A\, \text{adj}(A) \;=\; \det(A) I_n.$$

Consequently, if A is invertible, then $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$.

Proof.

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Consequently, if A is invertible, then $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$.

Proof. Let us first show that the first statement implies the second. Indeed, if A is invertible, then $\det(A) \neq 0$, and so if the first statement holds, then we get that

$$\left(\tfrac{1}{\det(A)} \text{adj}(A)\right) A \;=\; A \left(\tfrac{1}{\det(A)} \text{adj}(A)\right) \;=\; I_n,$$

and consequently, $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$.

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and consequently, $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$.

It remains to prove the first statement, i.e. that $\text{adj}(A)\, A = A\, \text{adj}(A) = \det(A) I_n$. We will prove that $\text{adj}(A)\, A = \det(A) I_n$; the proof of $A\, \text{adj}(A) = \det(A) I_n$ is in the Lecture Notes.

Proof (continued). Reminder: WTS $\text{adj}(A)\, A = \det(A) I_n$.

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We will prove this by showing that the matrices $\mathrm{adj}(A)\,A$ and $\det(A) I_n$ have the same corresponding entries.

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We will prove this by showing that the matrices $\text{adj}(A)\,A$ and $\det(A)I_n$ have the same corresponding entries. Fix indices $i, j \in \{1, \ldots, n\}$.

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We will prove this by showing that the matrices $\text{adj}(A)\, A$ and $\det(A) I_n$ have the same corresponding entries. Fix indices $i, j \in \{1, \ldots, n\}$. The i, j-th entry of the matrix $\det(A) I_n$ is $\det(A)$ if $i = j$, and is zero if $i \neq j$.

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We will prove this by showing that the matrices $\operatorname{adj}(A)\,A$ and $\det(A) I_n$ have the same corresponding entries. Fix indices $i, j \in \{1, \ldots, n\}$. The i, j-th entry of the matrix $\det(A) I_n$ is $\det(A)$ if $i = j$, and is zero if $i \neq j$. We must show this holds for the i, j-th entry of the matrices $\operatorname{adj}(A)\,A$ as well.

The i-th row of $\operatorname{adj}(A)$ is $\begin{bmatrix} C_{1,i} & \ldots & C_{n,i} \end{bmatrix}$, and the j-th column of A is $\begin{bmatrix} a_{1,j} & \ldots & a_{n,j} \end{bmatrix}^T$.

Proof (continued). Reminder: WTS $\text{adj}(A)\,A = \det(A)I_n$.

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Now, let B_1 be the matrix obtained by replacing the i-th column of A by the j-th column of A. Then $\det(B_1) = \sum_{k=1}^{n} a_{k,j} C_{k,i}$ (via Laplace expansion along the i-th column of B_1). But if $i = j$, then $\det(B_1) = \det(A)$ (because $B_1 = A$), and if $i \neq j$, then $\det(B_1) = 0$ (because B_1 has two identical columns, namely, the i-th and j-th column). $\square$

Theorem 7.8.2

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$ ($n \geq 2$). Then

$$\mathrm{adj}(A)\, A \;=\; A\, \mathrm{adj}(A) \;=\; \det(A) I_n.$$

Consequently, if A is invertible, then $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$.

Example 7.8.3

Show that the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix},$$

(with entries understood to be in $\mathbb{R}$) is invertible, and using Theorem 7.8.2, find its inverse A^{-1}.

Solution.

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Solution. The matrix A is upper triangular, and so its determinant can be computed by multiplying the entries along the main diagonal. So, $\det(A) = 1 \cdot 2 \cdot 3 = 6$. Since $\det(A) \neq 0$, Theorem 7.4.1 guarantees that A is invertible.

Solution (continued). Reminder: $\det(A) = 6$, A is invertible.

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In Example 7.8.1, we compute the adjugate matrix of A:

$$\text{adj}(A) = \begin{bmatrix} 6 & -3 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix}.$$

So, by Theorem 7.8.5, we have that

$$A^{-1} = \tfrac{1}{\det(A)}\text{adj}(A) = \tfrac{1}{6}\begin{bmatrix} 6 & -3 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & -1/3 \\ 0 & 0 & 1/3 \end{bmatrix}.$$

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Corollary 7.8.4

Let $\mathbb{F}$ be a field, and let $a, b, c, d \in \mathbb{F}$. Then the matrix

$$A \;=\; \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad \neq bc$, and in this case, the inverse of A is given by the formula

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Proof (outline).

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Proof (outline). It is easy to see that

$$\det(A) = ad - bc \qquad \text{and} \qquad \text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

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We know that A is invertible iff $\det(A) \neq 0$, which happens precisely when $ad \neq bc$. In this case, Theorem 7.8.2 guarantees that

$$A^{-1} = \tfrac{1}{\det(A)}\text{adj}(A) = \tfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

which is what we needed to show. □