Linear Algebra 2

Lecture #14

Scalar (inner) products

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- **Terminology:** Vector spaces over $\mathbb{R}$ are called *real vector spaces*, and vector spaces over $\mathbb{C}$ are called *complex vector spaces*.

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Definition

A *scalar product* (also called *inner product*) in a **real** vector space V is a function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ that satisfies the following four axioms:

r.1. for all $\mathbf{x} \in V$, $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, and equality holds iff $\mathbf{x} = \mathbf{0}$;

r.2. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$;

r.3. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{R}$, $\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha\langle \mathbf{x}, \mathbf{y} \rangle$;

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- The name "scalar product" comes from the fact that we multiply two vectors and obtain a scalar as a result.

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- Axioms r.2 and r.3 guarantee that the scalar product in a real vector space V is linear in the first variable (when we keep the second variable fixed).

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- Axioms r.2 and r.3 guarantee that the scalar product in a real vector space V is linear in the first variable (when we keep the second variable fixed).
- But in fact, axioms r.2, r.3, and r.4 guarantee that it is linear in the second variable as well (when we keep the first variable fixed).
  - More precisely, we have the following (next slide):

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Proof of r.2'.

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Proof of r.2'. For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, we have the following:

$$\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle \overset{\text{r.4}}{=} \langle \mathbf{y} + \mathbf{z}, \mathbf{x} \rangle \overset{\text{r.2}}{=} \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{z}, \mathbf{x} \rangle \overset{\text{r.4}}{=} \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle.$$

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Proof of r.3'. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{R}$, we have the following:

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- For example, for vectors $\begin{bmatrix} 1 & -2 & 5 \end{bmatrix}^T$ and $\begin{bmatrix} -3 & 2 & 1 \end{bmatrix}^T$ in $\mathbb{R}^3$, we compute:

$$\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \quad = \quad 1 \cdot (-3) + (-2) \cdot 2 + 5 \cdot 1 \quad = \quad -2.$$

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- We still need to check that $\cdot$ really is a scalar product, i.e. that it satisfies axioms r.1-r.4.
  - Later!

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$$\mathbf{x}^T \mathbf{y} \quad = \quad \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad = \quad \begin{bmatrix} \sum_{i=1}^{n} x_i y_i \end{bmatrix} \quad = \quad \begin{bmatrix} \mathbf{x} \cdot \mathbf{y} \end{bmatrix} .$$

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- So, if we identify 1×1 matrices with scalars, then we simply get that

$$\mathbf{x} \cdot \mathbf{y} \quad = \quad \mathbf{x}^T \mathbf{y} \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n .$$

Proposition 6.1.1

The standard scalar product in $\mathbb{R}^n$ is a scalar product.

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Proof. We need to check that the standard scalar product $\cdot$ in $\mathbb{R}^n$ satisfies the four axioms from the definition of a scalar product in a real vector space.

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Proof. We need to check that the standard scalar product $\cdot$ in $\mathbb{R}^n$ satisfies the four axioms from the definition of a scalar product in a real vector space.

r.1. For a vector $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ in $\mathbb{R}^n$, we have that

$$\mathbf{x} \cdot \mathbf{x} \ = \ \sum_{i=1}^{n} x_i^2 \ \stackrel{(*)}{\geq} \ 0,$$

and (*) is an equality iff $x_1 = \dots = x_n = 0$, i.e. iff $\mathbf{x} = \mathbf{0}$.

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Proof (continued). r.2. For vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$, $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$, and $\mathbf{z} = \begin{bmatrix} z_1 & \dots & z_n \end{bmatrix}^T$ in $\mathbb{R}^n$, we have that

$$\begin{aligned} (\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} &= \sum_{i=1}^{n} (x_i + y_i) z_i \\ &= \Big(\sum_{i=1}^{n} x_i z_i \Big) + \Big(\sum_{i=1}^{n} y_i z_i \Big) \\ &= \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z}. \end{aligned}$$

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Proof (continued). r.3. For vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$ in $\mathbb{R}^n$ and a scalar $\alpha \in \mathbb{R}$, we have that

$$(\alpha \mathbf{x}) \cdot \mathbf{y} \;=\; \sum_{i=1}^{n} (\alpha x_i) y_i \;=\; \alpha \sum_{i=1}^{n} x_i y_i \;=\; \alpha (\mathbf{x} \cdot \mathbf{y}).$$

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This proves that the standard scalar product in $\mathbb{R}^n$ really is a scalar product. $\square$

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- Indeed, for matrices $A = \begin{bmatrix} a_{i,j} \end{bmatrix}_{n \times m}$ and $B = \begin{bmatrix} b_{i,j} \end{bmatrix}_{n \times m}$ in $\mathbb{R}^{n \times m}$, we can define

$$\langle A, B \rangle \quad = \quad \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} b_{ij} .$$

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- A similar type of scalar product can be defined for matrices.
- Indeed, for matrices $A = \begin{bmatrix} a_{i,j} \end{bmatrix}_{n \times m}$ and $B = \begin{bmatrix} b_{i,j} \end{bmatrix}_{n \times m}$ in $\mathbb{R}^{n \times m}$, we can define

$$\langle A, B \rangle \quad = \quad \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} b_{ij} .$$

- It is easy to verify that this really is a scalar product in $\mathbb{R}^{n \times m}$ (the proof is similar to that of Proposition 6.1.1).

Definition

The *standard scalar product* of vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$ in $\mathbb{R}^n$ is given by

$$\mathbf{x} \cdot \mathbf{y} \quad := \quad \sum_{i=1}^{n} x_i y_i .$$

Proposition 6.1.1

The standard scalar product in $\mathbb{R}^n$ is a scalar product.

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Proposition 6.1.1

The standard scalar product in $\mathbb{R}^n$ is a scalar product.

- **Remark:** The standard scalar product is only one of many possible scalar products in $\mathbb{R}^n$.
  - A full characterization of all possible scalar products in $\mathbb{R}^n$ will be given in a later lecture (in a couple of months).

- If you know calculus, here is an example with integrals:

- If you know calculus, here is an example with integrals:

Proposition 6.1.2

Let $a, b \in \mathbb{R}$ be such that $a < b$, and let $\mathcal{C}_{[a,b]}$ be the (real) vector space of all continuous functions from the closed interval $[a, b]$ to $\mathbb{R}$. Then the function $\langle \cdot, \cdot \rangle : \mathcal{C}_{[a,b]} \times \mathcal{C}_{[a,b]} \to \mathbb{R}$ defined by

$$\langle f, g \rangle \quad := \quad \int_a^b f(x)g(x)dx$$

for all $f, g \in \mathcal{C}_{[a,b]}$ is a scalar product.

- Proof: Lecture Notes (optional).

Definition

A *scalar product* (also called *inner product*) in a **complex** vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ that satisfies the following four axioms:

c.1. for all $\mathbf{x} \in V$, $\langle \mathbf{x}, \mathbf{x} \rangle$ is a real number, $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, and equality holds iff $\mathbf{x} = \mathbf{0}$;

c.2. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$;

c.3. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{C}$, $\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$;

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- Axioms c.2 and c.3 guarantee that the scalar product in a complex vector space V is linear in the first variable (when we keep the second variable fixed).

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- Axioms c.2 and c.3 guarantee that the scalar product in a complex vector space V is linear in the first variable (when we keep the second variable fixed).
- Unlike in the real case, it is **not** linear in the second variable (when we keep the first variable fixed).
  - We do, however, have the following (next slide):

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c.3. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{C}$, $\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha\langle \mathbf{x}, \mathbf{y} \rangle$;

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c.2'. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$;

c.3'. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{C}$, $\langle \mathbf{x}, \alpha\mathbf{y} \rangle = \overline{\alpha}\langle \mathbf{x}, \mathbf{y} \rangle$.

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Proof.

c.2'. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x}, \mathbf{y}+\mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$;

c.3'. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{C}$, $\langle \mathbf{x}, \alpha\mathbf{y} \rangle = \overline{\alpha} \langle \mathbf{x}, \mathbf{y} \rangle$.

Proof. c.2'. For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, we have the following:

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y}+\mathbf{z} \rangle &\overset{c.4}{=} \overline{\langle \mathbf{y}+\mathbf{z}, \mathbf{x} \rangle} \\ &\overset{c.2}{=} \overline{\langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{z}, \mathbf{x} \rangle} \\ &= \overline{\langle \mathbf{y}, \mathbf{x} \rangle} + \overline{\langle \mathbf{z}, \mathbf{x} \rangle} \\ &\overset{c.4}{=} \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle. \end{aligned}$$

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\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle &\overset{\text{c.4}}{=} \overline{\langle \mathbf{y} + \mathbf{z}, \mathbf{x} \rangle} \\
&\overset{\text{c.2}}{=} \overline{\langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{z}, \mathbf{x} \rangle} \\
&= \overline{\langle \mathbf{y}, \mathbf{x} \rangle} + \overline{\langle \mathbf{z}, \mathbf{x} \rangle} \\
&\overset{\text{c.4}}{=} \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle.
\end{aligned}$$

c.3'. For all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{C}$, we have the following:

$$\langle \mathbf{x}, \alpha \mathbf{y} \rangle \overset{\text{c.4}}{=} \overline{\langle \alpha \mathbf{y}, \mathbf{x} \rangle} \overset{\text{c.3}}{=} \overline{\alpha \langle \mathbf{y}, \mathbf{x} \rangle} = \overline{\alpha}\, \overline{\langle \mathbf{y}, \mathbf{x} \rangle} \overset{\text{c.4}}{=} \overline{\alpha} \langle \mathbf{x}, \mathbf{y} \rangle.$$

□

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$$\mathbf{x} \cdot \mathbf{y} \quad := \quad \sum_{i=1}^{n} x_i \overline{y_i}.$$

- For example, for vectors $\begin{bmatrix} 1-2i & -2+i \end{bmatrix}^T$ and $\begin{bmatrix} 2+i & 1+3i \end{bmatrix}^T$ in $\mathbb{C}^2$, we compute:

$$\begin{aligned} \begin{bmatrix} 1-2i \\ -2+i \end{bmatrix} \cdot \begin{bmatrix} 2+i \\ 1+3i \end{bmatrix} &= (1-2i)\overline{(2+i)} + (-2+i)\overline{(1+3i)} \\ &= (1-2i)(2-i) + (-2+i)(1-3i) \\ &= 1+2i. \end{aligned}$$

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$$\mathbf{x} \cdot \mathbf{y} \quad := \quad \sum_{i=1}^{n} x_i \overline{y_i}.$$

Proposition 6.1.3

The standard scalar product in $\mathbb{C}^n$ is a scalar product.

- Proof: Lecture Notes (similar to the real case).

2. Orthogonality

2. Orthogonality

Definition

Given a real or complex vector space V, equipped with a scalar product $\langle \cdot, \cdot \rangle$, we say that vectors $\mathbf{x}$ and $\mathbf{y}$ in V are *orthogonal*, and we write $\mathbf{x} \perp \mathbf{y}$, if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

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- When our scalar product is the **standard** scalar product in $\mathbb{R}^n$, this corresponds to the usual geometric interpretation.
  - Details: Later!
- However, for general scalar products, this is how we **define** orthogonality.
  - For example, for the scalar product defined on $\mathcal{C}_{[-\pi,\pi]}$ in Proposition 6.1.2 (the one with integrals), we have that

$$\sin x \perp \cos x,$$

since $\left\langle \sin x, \cos x \right\rangle = \int_{-\pi}^{\pi} \sin x \cos x dx = 0.$

Proposition 6.1.4

Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Then all the following hold:

(a) for all vectors $\mathbf{x}, \mathbf{y} \in V$, we have that $\mathbf{x} \perp \mathbf{y}$ iff $\mathbf{y} \perp \mathbf{x}$;

(b) for all vectors $\mathbf{x}, \mathbf{y} \in V$ and scalars α, β,[a] if $\mathbf{x} \perp \mathbf{y}$ then $(\alpha\mathbf{x}) \perp (\beta\mathbf{y})$;

(c) for all vectors $\mathbf{x} \in V$, we have that $\mathbf{x} \perp \mathbf{0}$ and $\mathbf{0} \perp \mathbf{x}$.

[a] Here, α and β are real or complex numbers, depending on whether V is a real or complex vector space.

Proof.

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[a] Here, α and β are real or complex numbers, depending on whether V is a real or complex vector space.

Proof. We prove the proposition for the case when V is a complex vector space. The real case is similar but slightly easier (because we do not have to deal with complex conjugates).

Proposition 6.1.4

(a) for all vectors $\mathbf{x}, \mathbf{y} \in V$, we have that $\mathbf{x} \perp \mathbf{y}$ iff $\mathbf{y} \perp \mathbf{x}$

Proof (continued). (a) For vectors $\mathbf{x}, \mathbf{y} \in V$, we have the following sequence of equivalences:

$$
\begin{aligned}
\mathbf{x} \perp \mathbf{y} \quad &\iff \quad \langle \mathbf{x}, \mathbf{y} \rangle = 0 && \text{by definition} \\
&\iff \quad \overline{\langle \mathbf{y}, \mathbf{x} \rangle} = 0 && \text{by c.4} \\
&\iff \quad \langle \mathbf{y}, \mathbf{x} \rangle = 0 && \\
&\iff \quad \mathbf{y} \perp \mathbf{x} && \text{by definition.}
\end{aligned}
$$

Proposition 6.1.4

(b) for all vectors $\mathbf{x}, \mathbf{y} \in V$ and scalars α, β, if $\mathbf{x} \perp \mathbf{y}$ then $(\alpha\mathbf{x}) \perp (\beta\mathbf{y})$

Proof (continued). (b) Fix vectors $\mathbf{x}, \mathbf{y} \in V$ and scalars $\alpha, \beta \in \mathbb{C}$, and assume that $\mathbf{x} \perp \mathbf{y}$. Then we compute:

$$
\begin{aligned}
\langle \alpha\mathbf{x}, \beta\mathbf{y} \rangle &= \alpha\langle \mathbf{x}, \beta\mathbf{y} \rangle && \text{by c.3} \\
&= \alpha\overline{\beta}\langle \mathbf{x}, \mathbf{y} \rangle && \text{by c.3'} \\
&= \alpha\overline{\beta}0 && \text{beause } \mathbf{x} \perp \mathbf{y} \\
&= 0.
\end{aligned}
$$

So, $(\alpha\mathbf{x}) \perp (\beta\mathbf{y})$.

Proposition 6.1.4

Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Then all the following hold:

- (a) for all vectors $\mathbf{x}, \mathbf{y} \in V$, we have that $\mathbf{x} \perp \mathbf{y}$ iff $\mathbf{y} \perp \mathbf{x}$;
- (b) for all vectors $\mathbf{x}, \mathbf{y} \in V$ and scalars α, β,[a] if $\mathbf{x} \perp \mathbf{y}$ then $(\alpha \mathbf{x}) \perp (\beta \mathbf{y})$;
- (c) for all vectors $\mathbf{x} \in V$, we have that $\mathbf{x} \perp \mathbf{0}$ and $\mathbf{0} \perp \mathbf{x}$.

[a] Here, α and β are real or complex numbers, depending on whether V is a real or complex vector space.

Proof (continued). (c) Fix any vector $\mathbf{x} \in V$. We then have that

$$\langle \mathbf{0}, \mathbf{x} \rangle \;=\; \langle 0\mathbf{0}, \mathbf{x} \rangle \;\stackrel{\text{c.3}}{=}\; 0\langle \mathbf{0}, \mathbf{x} \rangle \;=\; 0,$$

and so $\mathbf{0} \perp \mathbf{x}$. The fact that $\mathbf{x} \perp \mathbf{0}$ now follows from (a). $\square$

Definition

Given a real or complex vector space V, equipped with a scalar product $\langle \cdot, \cdot \rangle$, we say that vectors $\mathbf{x}$ and $\mathbf{y}$ in V are *orthogonal*, and we write $\mathbf{x} \perp \mathbf{y}$, if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

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- Suppose that V is a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$.

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- Suppose that V is a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$.
  - For a vector $\mathbf{v} \in V$ and a set of vectors $A \subseteq V$, we say that $\mathbf{v}$ is *orthogonal* to A, and we write $\mathbf{v} \perp A$, provided that $\mathbf{v}$ is orthogonal to all vectors in A.
    - By definition, this means that for all $\mathbf{a} \in A$, we have that $\langle \mathbf{v}, \mathbf{a} \rangle = 0$.

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- Suppose that V is a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$.
  - For a vector $\mathbf{v} \in V$ and a set of vectors $A \subseteq V$, we say that $\mathbf{v}$ is *orthogonal* to A, and we write $\mathbf{v} \perp A$, provided that $\mathbf{v}$ is orthogonal to all vectors in A.
    - By definition, this means that for all $\mathbf{a} \in A$, we have that $\langle \mathbf{v}, \mathbf{a} \rangle = 0$.
  - For sets of vectors $A, B \subseteq V$, we say that A is *orthogonal* to B, and we write $A \perp B$, if every vector in A is orthogonal to every vector in B.

Proposition 6.1.5

Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\mathbf{a}_1, \dots, \mathbf{a}_p, \mathbf{b}_1, \dots, \mathbf{b}_q \in V$, and assume that $\{\mathbf{a}_1, \dots, \mathbf{a}_p\} \perp \{\mathbf{b}_1, \dots, \mathbf{b}_q\}$. Then $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_p) \perp \text{Span}(\mathbf{b}_1, \dots, \mathbf{b}_q)$.

Proof.

Proposition 6.1.5

Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\mathbf{a}_1, \dots, \mathbf{a}_p, \mathbf{b}_1, \dots, \mathbf{b}_q \in V$, and assume that $\{\mathbf{a}_1, \dots, \mathbf{a}_p\} \perp \{\mathbf{b}_1, \dots, \mathbf{b}_q\}$. Then $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_p) \perp \text{Span}(\mathbf{b}_1, \dots, \mathbf{b}_q)$.

Proof. Fix $\mathbf{a} \in \text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_p)$ and $\mathbf{b} \in \text{Span}(\mathbf{b}_1, \dots, \mathbf{b}_q)$. Then there exist scalars $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q$ s.t.

$$\mathbf{a} = \alpha_1 \mathbf{a}_1 + \dots + \alpha_p \mathbf{a}_p \qquad \text{and} \qquad \mathbf{b} = \beta_1 \mathbf{b}_1 + \dots + \beta_q \mathbf{b}_q.$$

We now compute (next slide):

Proof (continued).

$$\begin{aligned}
\langle \mathbf{a}, \mathbf{b} \rangle &= \Big\langle \sum_{i=1}^{p} \alpha_i \mathbf{a}_i, \sum_{j=1}^{q} \beta_j \mathbf{b}_j \Big\rangle \\
&= \sum_{i=1}^{p} \Big\langle \alpha_i \mathbf{a}_i, \sum_{j=1}^{q} \beta_j \mathbf{b}_j \Big\rangle \qquad \text{by r.2 or c.2} \\
&= \sum_{i=1}^{p} \sum_{j=1}^{q} \underbrace{\langle \alpha_i \mathbf{a}_i, \beta_j \mathbf{b}_j \rangle}_{\overset{(*)}{=} 0} \qquad \text{by r.2' or c.2'} \\
&= 0,
\end{aligned}$$

where (*) follows from Proposition 6.1.4(b) and from the fact that $\{\mathbf{a}_1, \dots, \mathbf{a}_p\} \perp \{\mathbf{b}_1, \dots, \mathbf{b}_q\}$. This proves that $\mathbf{a} \perp \mathbf{b}$, and the result follows. □

Proposition 6.1.5

Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\mathbf{a}_1, \dots, \mathbf{a}_p, \mathbf{b}_1, \dots, \mathbf{b}_q \in V$, and assume that $\{\mathbf{a}_1, \dots, \mathbf{a}_p\} \perp \{\mathbf{b}_1, \dots, \mathbf{b}_q\}$. Then $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_p) \perp \text{Span}(\mathbf{b}_1, \dots, \mathbf{b}_q)$.

3. The norm induced by a scalar product

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- The idea is that for a vector $\mathbf{x} \in V$, $||\mathbf{x}||$ is the distance from $\mathbf{x}$ to the origin, or alternatively, the length of the vector $\mathbf{x}$; $||\mathbf{x}||$ is always supposed to be a non-negative real number (even if V is a complex vector space).

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- For vectors $\mathbf{x}, \mathbf{y} \in V$, $||\mathbf{x} - \mathbf{y}||$ is supposed to be the distance between $\mathbf{x}$ and $\mathbf{y}$.
- Distance can be defined in a variety of ways.
- We first study norms induced by a scalar product.
- Later, we will define the norm in general and give some examples.

Definition

Given a scalar product $\langle \cdot, \cdot \rangle$ in a real or complex vector space V, we define the *norm in V induced by* $\langle \cdot, \cdot \rangle$ to be the function $|| \cdot || : V \to \mathbb{R}$ given by

$$||\mathbf{x}|| \quad := \quad \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \qquad \forall \mathbf{x} \in V.$$

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$$||\mathbf{x}|| \quad := \quad \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \qquad \forall \mathbf{x} \in V.$$

- In view of r.1 and c.1, for all $\mathbf{x} \in V$, we have that $||\mathbf{x}||$ is a non-negative **real** number, and moreover, $||\mathbf{x}|| = 0$ iff $\mathbf{x} = \mathbf{0}$.

Proposition 6.2.1

Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$. Then for all vectors $\mathbf{x} \in V$ and scalars α,[a] we have that

$$||\alpha \mathbf{x}|| = |\alpha| \, ||\mathbf{x}||.$$

[a] So, α is a real or complex number, depending on whether the vector space V is real or complex.

Proof.

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Proof. We consider only the complex case. The real case is similar but easier (because we do not have to deal with complex conjugates).

Proof (continued). So, assume that V is a complex vector space.

Proof (continued). So, assume that V is a complex vector space. Then for all vectors $\mathbf{x} \in V$ and scalars $\alpha \in \mathbb{C}$, we have that

$$\begin{aligned} ||\alpha\mathbf{x}|| &= \sqrt{\langle \alpha\mathbf{x}, \alpha\mathbf{x} \rangle} \\ &= \sqrt{\alpha\overline{\alpha}\langle \mathbf{x}, \mathbf{x} \rangle} && \text{by c.3 and c.3'} \\ &= \sqrt{|\alpha|^2\langle \mathbf{x}, \mathbf{x} \rangle} && \text{by Proposition 0.3.2} \\ &= |\alpha|\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \\ &= |\alpha|\ ||\mathbf{x}||. \end{aligned}$$

This completes the argument. □

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$$||\mathbf{x}|| \quad := \quad \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \qquad \forall \mathbf{x} \in V.$$

- Note that if $|| \cdot ||$ is the norm induced by the **standard** scalar product in $\mathbb{R}^n$, then for all vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ in $\mathbb{R}^n$, we have that

$$||\mathbf{x}|| \quad = \quad \sqrt{\mathbf{x} \cdot \mathbf{x}} \quad = \quad \sqrt{\sum_{i=1}^{n} x_i^2}.$$

So, we simply get the standard Euclidean length in $\mathbb{R}^n$.

- Suppose once again that $||\cdot||$ is the norm induced by the standard scalar product in $\mathbb{R}^n$.

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- It turns out that if $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$ are non-zero vectors in $\mathbb{R}^n$, then

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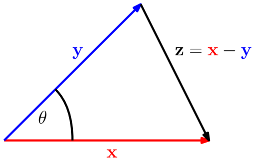
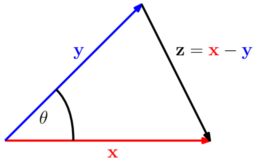
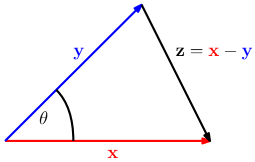
  where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$.
- Let us justify this!
- Consider the triangle formed by $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z} := \mathbf{x} - \mathbf{y}$, and let θ be the angle between $\mathbf{x}$ and $\mathbf{y}$ in this triangle.



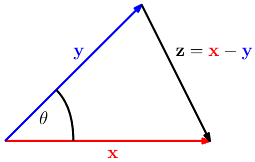


- We then compute:

$$
\begin{aligned}
||\mathbf{z}||^2 &= \mathbf{z}\cdot\mathbf{z} \\
&= (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) \\
&= \underbrace{\mathbf{x}\cdot\mathbf{x}}_{=||\mathbf{x}||^2} - \mathbf{x}\cdot\mathbf{y} - \mathbf{y}\cdot\mathbf{x} + \underbrace{\mathbf{y}\cdot\mathbf{y}}_{=||\mathbf{y}||^2} \\
&= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2\mathbf{x}\cdot\mathbf{y}
\end{aligned}
$$



- Reminder: $||\mathbf{z}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2\mathbf{x} \cdot \mathbf{y}$.



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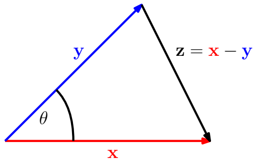
- On the other hand, the Law of Cosines (for triangles) tells us that

$$||\mathbf{z}||^2 \quad = \quad ||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2||\mathbf{x}|| \; ||\mathbf{y}|| \; \cos\theta.$$



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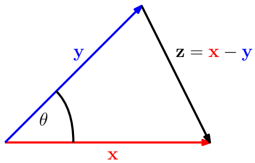
- So,

$$||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2\mathbf{x}\cdot\mathbf{y} = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2||\mathbf{x}||\ ||\mathbf{y}||\ \cos\theta,$$

and consequently,

$$\mathbf{x}\cdot\mathbf{y} = ||\mathbf{x}||\ ||\mathbf{y}||\ \cos\theta,$$

as we had claimed.



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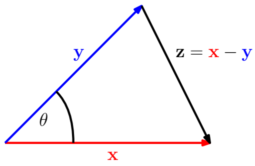

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- Note that this means that non-zero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are orthogonal in the usual geometric sense (i.e. the angle between them is 90°) iff $\mathbf{x} \cdot \mathbf{y} = 0$.
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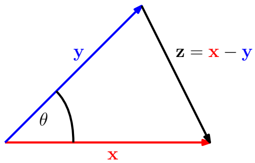

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- **Warning:** The formula $\mathbf{x} \cdot \mathbf{y} = ||\mathbf{x}|| \; ||\mathbf{y}|| \; \cos\theta$ works for the **standard** scalar product in $\mathbb{R}^n$ and the norm induced by it. Do not attempt to use it for general scalar products!

4. The Pythagorean theorem, the Cauchy–Schwarz inequality, and the triangle inequality

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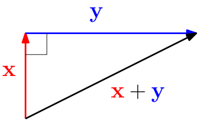
The Pythagorean theorem

Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $||\cdot||$. Then for all $\mathbf{x}, \mathbf{y} \in V$ such that $\mathbf{x} \perp \mathbf{y}$, we have that

$$||\mathbf{x} + \mathbf{y}||^2 \;\; = \;\; ||\mathbf{x}||^2 + ||\mathbf{y}||^2.$$



Proof.

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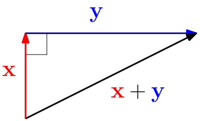

Proof. Fix $\mathbf{x}, \mathbf{y} \in V$ s.t. $\mathbf{x} \perp \mathbf{y}$.

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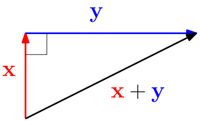
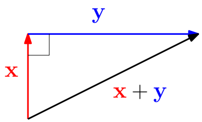


Proof. Fix $\mathbf{x}, \mathbf{y} \in V$ s.t. $\mathbf{x} \perp \mathbf{y}$. Then $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ and $\langle \mathbf{y}, \mathbf{x} \rangle = 0$.



Proof (continued). So,

$$
\begin{aligned}
||\mathbf{x}+\mathbf{y}||^2 &= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle \\
&= \underbrace{\langle \mathbf{x}, \mathbf{x}\rangle}_{=||\mathbf{x}||^2} + \underbrace{\langle \mathbf{x}, \mathbf{y}\rangle}_{=0} + \underbrace{\langle \mathbf{y}, \mathbf{x}\rangle}_{=0} + \underbrace{\langle \mathbf{y}, \mathbf{y}\rangle}_{=||\mathbf{y}||^2} \\
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which is what we needed to show. □

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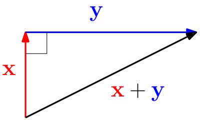

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$$|\langle \mathbf{x}, \mathbf{y} \rangle| \quad \leq \quad ||\mathbf{x}|| \, ||\mathbf{y}|| \qquad \forall \mathbf{x}, \mathbf{y} \in V.$$

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and we compute

$$\langle \mathbf{z}, \mathbf{y} \rangle \quad = \quad \langle \tfrac{\langle \mathbf{y}, \mathbf{y} \rangle}{\langle \mathbf{x}, \mathbf{y} \rangle}\mathbf{x} - \mathbf{y}, \mathbf{y} \rangle \quad \overset{(*)}{=} \quad \tfrac{\langle \mathbf{y}, \mathbf{y} \rangle}{\langle \mathbf{x}, \mathbf{y} \rangle}\langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{y}, \mathbf{y} \rangle \quad = \quad 0,$$

where (*) follows from r.2 and r.3 if V is a real vector space, or from c.2 and c.3 if V is a complex vector space.

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Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$. Then

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Proof. Fix $\mathbf{x}, \mathbf{y} \in V$. WMA $\langle \mathbf{x}, \mathbf{y} \rangle \neq 0$, for otherwise, the result is immediate. Note that this implies that $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$, and consequently, $||\mathbf{x}||, ||\mathbf{y}|| \neq 0$.

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where (*) follows from r.2 and r.3 if V is a real vector space, or from c.2 and c.3 if V is a complex vector space. So, $\mathbf{z} \perp \mathbf{y}$.

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Proof (continued). Reminder: $\mathbf{z} := \frac{\langle \mathbf{y}, \mathbf{y} \rangle}{\langle \mathbf{x}, \mathbf{y} \rangle} \mathbf{x} - \mathbf{y}$; $\mathbf{z} \perp \mathbf{y}$.

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Proof (continued). Reminder: $\mathbf{z} := \frac{\langle \mathbf{y}, \mathbf{y} \rangle}{\langle \mathbf{x}, \mathbf{y} \rangle}\mathbf{x} - \mathbf{y}$; $\mathbf{z} \perp \mathbf{y}$.

By the Pythagorean theorem, we have that $||\mathbf{z} + \mathbf{y}||^2 = ||\mathbf{z}||^2 + ||\mathbf{y}||^2$, and consequently:

$$||\mathbf{z} + \mathbf{y}|| \quad = \quad ||\tfrac{\langle \mathbf{y}, \mathbf{y} \rangle}{\langle \mathbf{x}, \mathbf{y} \rangle}\mathbf{x}|| \quad \overset{(*)}{=} \quad |\tfrac{\langle \mathbf{y}, \mathbf{y} \rangle}{\langle \mathbf{x}, \mathbf{y} \rangle}| \; ||\mathbf{x}|| \quad = \quad \tfrac{|\langle \mathbf{y}, \mathbf{y} \rangle|}{|\langle \mathbf{x}, \mathbf{y} \rangle|} \; ||\mathbf{x}|| \quad = \quad \tfrac{||\mathbf{y}||^2}{|\langle \mathbf{x}, \mathbf{y} \rangle|} \; ||\mathbf{x}||,$$

where (*) follows from Proposition 6.2.1.

The Cauchy–Schwarz inequality

Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$. Then

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \quad \leq \quad ||\mathbf{x}|| \, ||\mathbf{y}|| \qquad \forall \mathbf{x}, \mathbf{y} \in V.$$

Proof (continued). Reminder: $\mathbf{z} := \frac{\langle \mathbf{y}, \mathbf{y} \rangle}{\langle \mathbf{x}, \mathbf{y} \rangle} \mathbf{x} - \mathbf{y}$; $\mathbf{z} \perp \mathbf{y}$.

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where (*) follows from Proposition 6.2.1. So,

$$\tfrac{||\mathbf{y}||^4}{|\langle \mathbf{x}, \mathbf{y} \rangle|^2} \, ||\mathbf{x}||^2 \quad = \quad ||\mathbf{z} + \mathbf{y}||^2 \quad = \quad ||\mathbf{z}||^2 + ||\mathbf{y}||^2 \quad \geq \quad ||\mathbf{y}||^2,$$

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Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$. Then

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which yields

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Let V be a real or complex vector space, equipped with a scalar product $\langle\cdot,\cdot\rangle$ and the induced norm $||\cdot||$. Then

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Proof (continued). Reminder: $\frac{||\mathbf{y}||^4}{|\langle \mathbf{x},\mathbf{y}\rangle|^2}\,||\mathbf{x}||^2 \geq ||\mathbf{y}||^2$.

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Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$. Then

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Proof (continued). Reminder: $\frac{||\mathbf{y}||^4}{|\langle \mathbf{x},\mathbf{y} \rangle|^2} \, ||\mathbf{x}||^2 \geq ||\mathbf{y}||^2$.

Since $\langle \mathbf{x}, \mathbf{y} \rangle$ and $||\mathbf{y}||$ are both non-zero, we have that $\frac{|\langle \mathbf{x},\mathbf{y} \rangle|^2}{||\mathbf{y}||^2}$ is defined and positive.

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$$||\mathbf{x}||^2||\mathbf{y}||^2 \quad \geq \quad |\langle \mathbf{x}, \mathbf{y} \rangle|^2.$$

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By taking the square root of both sides, we get

$$||\mathbf{x}|| \, ||\mathbf{y}|| \quad \geq \quad |\langle \mathbf{x}, \mathbf{y} \rangle|,$$

which is what we needed to show. □

The Cauchy–Schwarz inequality

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Corollary 6.2.2

For all $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$, we have that

$$\Big(\sum_{i=1}^{n} x_i y_i\Big)^2 \quad \leq \quad \Big(\sum_{i=1}^{n} x_i^2\Big)\Big(\sum_{i=1}^{n} y_i^2\Big).$$

Proof.

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Proof. If we consider the standard scalar product in $\mathbb{R}^n$, the Cauchy-Schwarz inequality yields

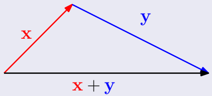
$$\Big| \sum_{i=1}^{n} x_i y_i \Big| \quad \leq \quad \sqrt{\sum_{i=1}^{n} x_i^2} \sqrt{\sum_{i=1}^{n} y_i^2}.$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$. By squaring both sides, we obtain the desired inequality. $\square$

The triangle inequality

Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$. Then

$$||\mathbf{x} + \mathbf{y}|| \quad \leq \quad ||\mathbf{x}|| + ||\mathbf{y}|| \qquad \forall \mathbf{x}, \mathbf{y} \in V.$$



Proof.

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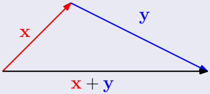

Proof. We prove the result for the case when V is a complex vector space. The real case is similar but slightly easier (because we do not have to deal with complex conjugates).

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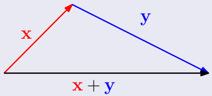

Proof. We prove the result for the case when V is a complex vector space. The real case is similar but slightly easier (because we do not have to deal with complex conjugates).

We first remark that for all complex numbers $z = a + ib$ (where $a, b \in \mathbb{R}$), we have that

- $z + \overline{z} = 2a = 2\mathrm{Re}(z)$;
- $\mathrm{Re}(z) = a \leq |a| \leq \sqrt{a^2 + b^2} = |z|$.

Proof (continued). Now, fix $\mathbf{x}, \mathbf{y} \in V$.

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$$\begin{array}{rcll}
||\mathbf{x}+\mathbf{y}||^2 & = & \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle & \\
 & = & \underbrace{\langle \mathbf{x}, \mathbf{x} \rangle}_{=||\mathbf{x}||^2} + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \underbrace{\langle \mathbf{y}, \mathbf{y} \rangle}_{=||\mathbf{y}||^2} & \text{by c.2 and c.2'} \\
 & = & ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle & \\
 & = & ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + \langle \mathbf{x}, \mathbf{y} \rangle + \overline{\langle \mathbf{x}, \mathbf{y} \rangle} & \text{by c.4} \\
 & = & ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2\mathrm{Re}\big(\langle \mathbf{x}, \mathbf{y} \rangle\big) & \\
 & \leq & ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2|\langle \mathbf{x}, \mathbf{y} \rangle| & \\
 & \leq & ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2||\mathbf{x}||\,||\mathbf{y}|| & \text{by C-S}\leq \\
 & = & (||\mathbf{x}|| + ||\mathbf{y}||)^2. &
\end{array}$$

Proof (continued). Now, fix $\mathbf{x}, \mathbf{y} \in V$. Then we have the following:

$$\begin{aligned}
||\mathbf{x}+\mathbf{y}||^2 &= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle \\
&= \underbrace{\langle \mathbf{x}, \mathbf{x} \rangle}_{=||\mathbf{x}||^2} + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \underbrace{\langle \mathbf{y}, \mathbf{y} \rangle}_{=||\mathbf{y}||^2} && \text{by c.2 and c.2'} \\
&= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle \\
&= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + \langle \mathbf{x}, \mathbf{y} \rangle + \overline{\langle \mathbf{x}, \mathbf{y} \rangle} && \text{by c.4} \\
&= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2\mathrm{Re}\big(\langle \mathbf{x}, \mathbf{y} \rangle\big) \\
&\leq ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2|\langle \mathbf{x}, \mathbf{y} \rangle| \\
&\leq ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2||\mathbf{x}|| \, ||\mathbf{y}|| && \text{by C-S}\leq \\
&= (||\mathbf{x}|| + ||\mathbf{y}||)^2.
\end{aligned}$$

By taking the square root of both sides, we obtain $||\mathbf{x}+\mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$, which is what we needed to show. □

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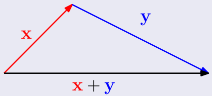

5. The norm in general

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Definition

A *norm* in a real or complex vector space V is a function $||\cdot|| : V \to \mathbb{R}$ that satisfies the following three axioms:

n.1. for all vectors $\mathbf{x} \in V$, we have that $||\mathbf{x}|| \geq 0$, and equality holds iff $\mathbf{x} = \mathbf{0}$;

n.2. for all vectors $\mathbf{x} \in V$ and scalars α,[a] we have that $||\alpha\mathbf{x}|| = |\alpha|\ ||\mathbf{x}||$;

n.3. for all vectors $\mathbf{x}, \mathbf{y} \in V$, we have that $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$.

---

[a] So, α is a real or complex number, depending on whether the vector space V is real or complex.

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n.3. for all vectors $\mathbf{x}, \mathbf{y} \in V$, we have that $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$.

- A norm in a real or complex vector space V gives a way of measuring the distance of a vector from the origin, or equivalently, measuring the length of a vector.
  - The norm of a vector is always a non-negative real number (regardless of whether the vector space is real or complex).

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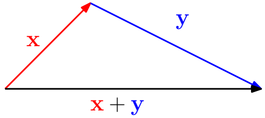
n.3. for all vectors $\mathbf{x}, \mathbf{y} \in V$, we have that $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$.

- We note that n.3 is referred to as the "triangle inequality."



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- Any norm induced by a scalar product in a real or complex vector space V really is a norm, i.e. it is a function from V to $\mathbb{R}$ that satisfies axioms n.1, n.2, and n.3 above.

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- Any norm induced by a scalar product in a real or complex vector space V really is a norm, i.e. it is a function from V to $\mathbb{R}$ that satisfies axioms n.1, n.2, and n.3 above.
  - The fact that axiom n.1 is satisfied is immediate from the construction of a norm induced by a scalar product, the fact that n.2 is satisfied follows from Proposition **??**, and the fact that n.3 is satisfied follows from the triangle inequality proven a few slides ago.

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Let V be a real or complex vector space, equipped with a norm $||\cdot||$. A vector $\mathbf{v} \in V$ is called a *unit vector* if $||\mathbf{v}|| = 1$.

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Let V be a real or complex vector space, equipped with a norm $||\cdot||$. A vector $\mathbf{v} \in V$ is called a *unit vector* if $||\mathbf{v}|| = 1$.

- By n.1, any unit vector is, in particular, a non-zero vector.

Definition

A *norm* in a real or complex vector space V is a function $|| \cdot || : V \to \mathbb{R}$ that satisfies the following three axioms:

n.1. for all vectors $\mathbf{x} \in V$, we have that $||\mathbf{x}|| \geq 0$, and equality holds iff $\mathbf{x} = \mathbf{0}$;

n.2. for all vectors $\mathbf{x} \in V$ and scalars α, we have that $||\alpha\mathbf{x}|| = |\alpha| \; ||\mathbf{x}||$;

n.3. for all vectors $\mathbf{x}, \mathbf{y} \in V$, we have that $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$.

Definition

Let V be a real or complex vector space, equipped with a norm $|| \cdot ||$. A vector $\mathbf{v} \in V$ is called a *unit vector* if $||\mathbf{v}|| = 1$.

- By n.1, any unit vector is, in particular, a non-zero vector.
- For notational convenience, given a vector $\mathbf{v}$ and a scalar $\alpha \neq 0$, we often write $\frac{\mathbf{v}}{\alpha}$ instead of $\alpha^{-1}\mathbf{v}$ or $\frac{1}{\alpha}\mathbf{v}$.
  - In particular, for a non-zero vector $\mathbf{v} \in V$, we may write $\frac{\mathbf{v}}{||\mathbf{v}||}$.

Proposition 6.2.3

Let V be a real or complex vector space, equipped with a norm $||\cdot||$. Then for all non-zero vectors $\mathbf{v} \in V$, we have that $||\mathbf{v}|| > 0$ and that $||\frac{\mathbf{v}}{||\mathbf{v}||}|| = 1$, and in particular, $\frac{\mathbf{v}}{||\mathbf{v}||}$ is a unit vector.

Proof.

Proposition 6.2.3

Let V be a real or complex vector space, equipped with a norm $||\cdot||$. Then for all non-zero vectors $\mathbf{v} \in V$, we have that $||\mathbf{v}|| > 0$ and that $||\frac{\mathbf{v}}{||\mathbf{v}||}|| = 1$, and in particular, $\frac{\mathbf{v}}{||\mathbf{v}||}$ is a unit vector.

Proof. Fix a non-zero vector $\mathbf{v} \in V$. By n.1, we have that $||\mathbf{v}|| > 0$. We further have that

$$||\tfrac{\mathbf{v}}{||\mathbf{v}||}|| \overset{\text{n.2}}{=} |\tfrac{1}{||\mathbf{v}||}|\ ||\mathbf{v}|| \overset{(*)}{=} \tfrac{1}{||\mathbf{v}||}\ ||\mathbf{v}|| = 1,$$

where (*) follows from the fact that $||\mathbf{v}|| > 0$. This completes the argument. □

Proposition 6.2.3

Let V be a real or complex vector space, equipped with a norm $|| \cdot ||$. Then for all non-zero vectors $\mathbf{v} \in V$, we have that $||\mathbf{v}|| > 0$ and that $||\frac{\mathbf{v}}{||\mathbf{v}||}|| = 1$, and in particular, $\frac{\mathbf{v}}{||\mathbf{v}||}$ is a unit vector.

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- **Terminology/Remark:** Suppose that V is a real or complex vector space, equipped with a norm $||\cdot||$.

Proposition 6.2.3

Let V be a real or complex vector space, equipped with a norm $||\cdot||$. Then for all non-zero vectors $\mathbf{v} \in V$, we have that $||\mathbf{v}|| > 0$ and that $||\frac{\mathbf{v}}{||\mathbf{v}||}|| = 1$, and in particular, $\frac{\mathbf{v}}{||\mathbf{v}||}$ is a unit vector.

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  - To *normalize* a non-zero vector $\mathbf{v}$ in V means to multiply that vector by $\frac{1}{||\mathbf{v}||}$ ("divide by its length").
  - By Proposition 6.2.3, when we normalize a non-zero vector, we produce a unit vector.

Definition

A *norm* in a real or complex vector space V is a function $||\cdot|| : V \to \mathbb{R}$ that satisfies the following three axioms:

n.1. for all vectors $\mathbf{x} \in V$, we have that $||\mathbf{x}|| \geq 0$, and equality holds iff $\mathbf{x} = \mathbf{0}$;

n.2. for all vectors $\mathbf{x} \in V$ and scalars α, we have that $||\alpha\mathbf{x}|| = |\alpha|\ ||\mathbf{x}||$;

n.3. for all vectors $\mathbf{x}, \mathbf{y} \in V$, we have that $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$.

- We now consider a few examples of norms in real vector spaces.

Definition

For a positive integer p, we define the *p-norm* in $\mathbb{R}^n$, denoted by $||\cdot||_p$, by setting

$$||\mathbf{x}||_p \quad := \quad \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

for all $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ in $\mathbb{R}^n$.

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- We omit the proof of the fact that this really is a norm in $\mathbb{R}^n$.
- We do note, however, that for $p = 2$, we get

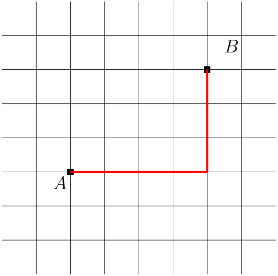
$$||\mathbf{x}||_2 \;=\; \sqrt{\sum_{i=1}^{n} x_i^2},$$

which is precisely the norm induced by the standard scalar product in $\mathbb{R}^n$, i.e. the standard Euclidean norm in $\mathbb{R}^n$.

- Reminder: $||\mathbf{x}||_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \; \forall \mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T \in \mathbb{R}^n$.
- For $p = 1$, we get

$$||\mathbf{x}||_1 = \sum_{i=1}^{n} |x_i|.$$

- The $|| \cdot ||_1$ norm is sometimes called the "Manhattan norm." This is because streets and avenues in Manhattan form a perfect grid (more or less), and so $|| \cdot ||_1$ gives the actual walking distance between two places in Manhattan.



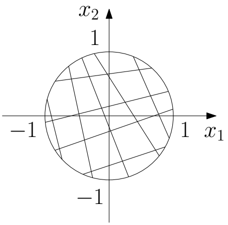
- Another norm of interest is the so called "Chebyshev distance" in $\mathbb{R}^n$, denoted by $||\cdot||_\infty$. It is defined by

$$||\mathbf{x}||_\infty \quad := \quad \max\left\{|x_1|, \ldots, |x_n|\right\}$$

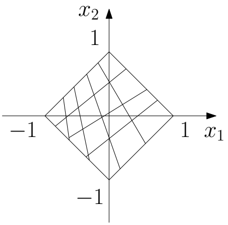
for all vectors $\mathbf{x} = \left[\begin{array}{ccc} x_1 & \ldots & x_n \end{array}\right]^T$ in $\mathbb{R}^n$.

Definition

The *unit disk* in a real or complex vector space V, equipped with a norm $||\cdot||$, is the set $\{\mathbf{x} \in V \mid ||\mathbf{x}|| \leq 1\}$.



unit disk in $\mathbb{R}^2$
w.r.t. $||\cdot||_2$



unit disk in $\mathbb{R}^2$
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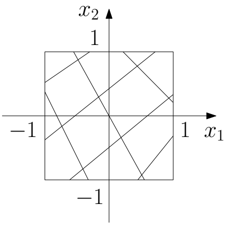

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- Finally, if you have studied calculus, recall that for $a, b \in \mathbb{R}$ such that $a < b$, $\mathcal{C}_{[a,b]}$ is the (real) vector space of all continuous functions from $[a, b]$ to $\mathbb{R}$.

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- For a real number $p \geq 1$, we have the norm $|| \cdot ||_p$ on $\mathcal{C}_{[a,b]}$ given by

$$||f||_p = \left(\int_a^b |f(x)|^p \right)^{\frac{1}{p}}$$

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- We also have the norm $|| \cdot ||_\infty$ on $\mathcal{C}_{[a,b]}$ given by

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- For a real number $p \geq 1$, we have the norm $|| \cdot ||_p$ on $\mathcal{C}_{[a,b]}$ given by

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  for all $f \in \mathcal{C}_{[a,b]}$.
- Once again, we omit the proof of the fact that $|| \cdot ||_p$ (for a real number $p \geq 1$) and $|| \cdot ||_\infty$ really are norms in $\mathcal{C}_{[a,b]}$.