Linear Algebra 2:
HW#8

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due Monday, May 20, 2024, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Exercise 1 (10 points)**.** *Suppose you are given a matrix* $A \in \mathbb{C}^{14\times 14}$ *for which your friend the calculator computes the characteristic polynomial*

$$p_A(\lambda) \quad = \quad (\lambda-2)^4(\lambda+3)^4(\lambda-4)^6,$$

and further helpfully computes the following:

- $rank\big((A-2I_{14})^0\big) = 14$;
- $rank\big((A-2I_{14})^1\big) = 12$;
- $rank\big((A-2I_{14})^2\big) = 11$;
- $rank\big((A-2I_{14})^3\big) = 10$;
- $rank\big((A-2I_{14})^4\big) = 10$;

- $rank\big((A+3I_{14})^0\big) = 14$;
- $rank\big((A+3I_{14})^1\big) = 12$;
- $rank\big((A+3I_{14})^2\big) = 10$;
- $rank\big((A+3I_{14})^3\big) = 10$;

- $rank\big((A-4I_{14})^0\big) = 14$;
- $rank\big((A-4I_{14})^1\big) = 11$;
- $rank\big((A-4I_{14})^2\big) = 10$;
- $rank\big((A-4I_{14})^3\big) = 9$;
- $rank\big((A-4I_{14})^4\big) = 8$;
- $rank\big((A-4I_{14})^5\big) = 8$.

Compute the Jordan normal form of A.

Problem 1 (15 points). *Consider all possible matrices in* $\mathbb{C}^{14\times 14}$ *whose eigenvalues are (exactly) the following:*

- $\lambda_1 = 3$ *with algebraic multipicity* 5 *and geometric multiplicity* 4*;*
- $\lambda_2 = 5$ *with algebraic multiplicity* 4 *and geometric multiplicity* 1*;*
- $\lambda_3 = 6$ *with algebraic multiplicity* 5 *and geometric multiplicity* 2*.*

Are all such matrices similar? Make sure you fully justify your answer.

> ***Hint:*** *Jordan normal form.*

Problem 2 (25 points). *Orthogonally diagonalize the following symmetric matrix in* $\mathbb{R}^{3\times 3}$*:*

$$A \;:=\; \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}.$$

In other words, compute a diagonal matrix D *and an orthogonal matrix* Q*, both in* $\mathbb{R}^{3\times 3}$*, such that* $D = Q^T A Q$*.*

> ***Hint:*** *This is very similar to Example 8.7.7 of the Lecture Notes.*

Problem 3 (25 points). *Let* $A \in \mathbb{R}^{n\times m}$*. Prove that there exists an orthonormal basis* $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ *of* $\mathbb{R}^m$ *such that vectors* $A\mathbf{v}_1, \dots, A\mathbf{v}_m$ *are pairwise orthogonal. (Here, orthogonality and orthonormality are assumed to be with respect to the standard scalar product* $\cdot$ *and the induced norm* $||\cdot||$*.)*

> ***Remark:*** *It is possible that some of the* $A\mathbf{v}_i$*'s are zero.*
>
> ***Hint:*** *Explain why* $\mathbb{R}^m$ *has an orthonormal eigenbasis associated with* $A^T A$*, and then use that eigenbasis.*

Problem 4 (25 points). *Prove or disprove the following statement:*

> *For every symmetric matrix* $A \in \mathbb{R}^{n\times n}$*, and all vectors* $\mathbf{v} \in$ *Col*(A) *and* $\mathbf{w} \in$ *Nul*(A)*, we have that* $\mathbf{v} \perp \mathbf{w}$*.*

(Here, orthogonality is assumed to be with respect to the standard scalar product $\cdot$ *in* $\mathbb{R}^n$*.) First, state clearly whether the statement is true or false. If it is true, then prove it. If it is false, then construct a counterexample (and prove that your counterexample really is a counterexample).*