Linear Algebra 2:
HW#7

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due Monday, May 6, 2024, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Exercise 1 (20 points)**.** *Consider the following matrices in* $\mathbb{C}^{6\times 6}$*:*

$$D = \begin{bmatrix} 7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}, \qquad P := \begin{bmatrix} 1 & 5 & -2 & 3 & 0 & 8 \\ 0 & 0 & 3 & -2 & 1 & -2 \\ -4 & 2 & 1 & 1 & 1 & 5 \\ 6 & -6 & -6 & 7 & 0 & 1 \\ 9 & 1 & 9 & 1 & 9 & 1 \\ 2 & 3 & 2 & 4 & 4 & 4 \end{bmatrix}.$$

You may assume that P is invertible.[1] *Set $A := PDP^{-1}$ (so, $D = P^{-1}AP$). Compute the characteristic polynomial of A and the spectrum of A. Identify the eigevalues of A, and for each eigenvalue, specify its algebraic and geometric multiplicity. Find a basis of each eigenspace of A.*

> ***Hint:*** *See Proposition 8.5.12 and Example 8.5.13 of the Lecture Notes.*

Problem 1 (20 points)**.** *For what values of complex constants a, b, c does $\mathbb{C}^3$ have an eigenbasis associated with the matrix A (below)? Make sure you prove that your answer is correct.*

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 2 \end{bmatrix}$$

[1] Indeed, the calculator tells us that $\det(P) = 4040 \neq 0$, and so P is invertible.

Problem 2 (20 points)**.** *Consider the following matrix in* $\mathbb{C}^{4\times 4}$*:*

$$A \;:=\; \begin{bmatrix} 20 & 0 & 9 & 0 \\ 0 & -1 & 0 & 0 \\ -42 & 0 & -19 & 0 \\ -21 & 0 & -9 & -1 \end{bmatrix}.$$

Diagonalize the matrix A*, and then compute a formula for* A^m*, where* m *is a non-negative integer. Does your formula also work for negative integers* m*? Why or why not?*

Problem 3 (20 points)**.** *Let* $\mathbb{F}$ *be a field, and let* $A \in \mathbb{F}^{n\times n}$ *be a square matrix such that* $\mathbb{F}^n$ *has an eigenbasis associated with* A*. Prove that* $\mathbb{F}^n$ *has an eigenbasis associated with* A^T*.*

> ***Hint:*** *Diagonalization.*

Problem 4 (20 points)**.** *In what follows,* $\mathbf{e}_1, \dots, \mathbf{e}_n$ *are the standard basis vectors of* $\mathbb{C}^n$*. Prove or disprove the following statement:*

> *If* $\mathbf{e}_1, \dots, \mathbf{e}_n$ *are eigenvectors of a matrix* $A \in \mathbb{C}^{n\times n}$*, then* A *must be diagonal.*

Remark: *First, state clearly whether the statement is true or false. If it is true, then prove it. If it is false, then construct a counterexample (and prove that your counterexample really is a counterexample).*