Linear Algebra 2:
HW#6

Todor Antić & Irena Penev
Summer 2024

due Monday, April 29, 2024, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Exercise 1 (20 points)**.** *Consider the following matrix in $\mathbb{C}^{4\times 4}$:*

$$A \quad := \quad \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

Compute the characteristic polynomial and the spectrum of A. For each eigenvalue λ of A, determine the algebraic and geometric multiplicity of λ, and compute a basis of the eigenspace $E_\lambda(A)$.

Problem 1 (20 points)**.** *Consider the linear function $f : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ given by*

$$f\big(a_2x^2 + a_1x + a_0\big) \quad = \quad (-9a_1 - 8a_2)x^2 + (7a_1 + 6a_2)x + (a_0 + 3a_1 + 2a_2)$$

for all $a_0, a_1, a_2 \in \mathbb{C}$. (You may assume that f is indeed linear, i.e. you do not need to prove that it is linear.) Compute the characteristic polynomial and the spectrum of f. For each eigenvalue λ of f, determine the algebraic and geometric multiplicity of λ, and compute a basis of the eigenspace $E_\lambda(f)$.

> ***Hint:*** *Imitate the solution of Example 8.2.16 from the Lecture Notes.*

Problem 2 (20 points)**.** *Consider the linear function* $f : \mathbb{C}^{2\times 2} \to \mathbb{C}^{2\times 2}$ *given by*

$$f\Big(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\Big) = \begin{bmatrix} a+d & b+c \\ b+c & a+d \end{bmatrix}$$

for all $a, b, c, d \in \mathbb{C}$*. (You may assume that* f *is indeed linear, i.e. you do not need to prove that it is linear.) Compute the characteristic polynomial and the spectrum of* f*. For each eigenvalue* λ *of* f*, determine the algebraic and geometric multiplicity of* λ*, and compute a basis of the eigenspace* $E_\lambda(f)$*.*

> ***Hint:*** *Once again, this is similar to Example 8.2.16 from the Lecture Notes, except that it involves matrices rather than polynomials.*

Problem 3 (20 points)**.** *Prove or disprove the following statement:*

> *For all matrices* $A, B \in \mathbb{C}^{2\times 2}$ *and vectors* $\mathbf{v} \in \mathbb{C}^2$*, if* $\mathbf{v}$ *is an eigenvector of both* A *and* B*, then* $\mathbf{v}$ *is an eigenvector of* $A + B$*.*

First, state clearly whether the statement is true or false. If it is true, then prove it. If it is false, then construct a counterexample (and prove that your counterexample really is a counterexample).

Problem 4 (20 points)**.** *Prove or disprove the following statement:*

> *For all matrices* $A, B \in \mathbb{C}^{2\times 2}$ *and scalars* $\lambda \in \mathbb{C}$*, if* λ *is an eigenvalue of both* A *and* B*, then* λ *is an eigenvalue of* $A + B$*.*

First, state clearly whether the statement is true or false. If it is true, then prove it. If it is false, then construct a counterexample (and prove that your counterexample really is a counterexample).