Linear Algebra 2: HW#5

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due Monday, April 22, 2024, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Exercise 1 (10 points)**.** *Consider the following matrix and vector, with entries understood to be in* $\mathbb{Z}_3$*:*

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Either use ***Cramer's rule*** *to solve the matrix-vector equation* $A\mathbf{x} = \mathbf{b}$*, or explain why this equation cannot be solved using Cramer's rule. (If the equation cannot be solved using Cramer's rule, then you do not need to solve it.)*

Exercise 2 (10 points)**.** *Compute the determinant of the following matrix (with entries understood to be in* $\mathbb{R}$*):*

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 & 2 \\ -3 & 2 & 1 & 0 & -1 \\ 0 & -2 & 1 & 0 & 0 \\ 1 & 3 & -1 & 2 & 3 \\ 2 & -1 & 1 & 0 & 0 \end{bmatrix}.$$

You may use any method (or combination of methods) that you like, but make sure you show your work.[1]

[1] You should compute this by hand (and show your work!), but you can, and probably should, check your answer with a calculator.

Exercise 3 (15 points). *Consider the matrix*

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

with entries understood to be in $\mathbb{R}$*. Compute:*

1. *det*(A)*;*
2. *the cofactor matrix of* A*;*
3. *the adjugate matrix of* A*.*

Is A *invertible? If so, compute its inverse.*

Problem 1 (15 points). *Let* a, b, c *be positive real numbers. Compute the volume of the solid enclosed by the ellipsoid*

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1, x_2, x_3 \in \mathbb{R},\ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1 \right\}.$$

> ***Hint:*** *Imitate the solution of Example 7.10.6 from the Lecture Notes.*

Problem 2 (25 points). *Let* $\mathbb{F}$ *be a field, let* $n \geq 2$ *be an integer, and let* $\mathbf{a}_2, \ldots, \mathbf{a}_n$ *be linearly independent vectors in* $\mathbb{F}^n$*. Define the function* $f : \mathbb{F}^n \to \mathbb{F}$ *by setting*

$$f(\mathbf{x}) = \begin{vmatrix} \mathbf{x} & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{vmatrix} \qquad \textit{for all } \mathbf{x} \in \mathbb{F}^n.$$

By Proposition 7.2.1 of the Lecture Notes, f *is linear. Find a basis of Ker*(f)*, and compute the dimension of Ker*(f) *and Im*(f)*. Make sure you fully justify your answer.*

Problem 3 (25 points). *Let* A *be an invertible matrix in* $\mathbb{R}^{n \times n}$*, and assume that the entries of* A *are all integers. Prove that the entries of* A^{-1} *are all integers if and only if det*$(A) = \pm 1$*.*

> ***Hint:*** *Use the adjugate matrix.*