# Linear Algebra 2:
# HW#4

Todor Antić & Irena Penev
Summer 2024

due Monday, April 8, 2024, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

**Exercise 1** (10 points)**.** *Consider the following matrix and vector, with entries understood to be in* $\mathbb{R}$*:*

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

*Find the least-squares solution(s) of the matrix-vector equation* $A\mathbf{x} = \mathbf{b}$*, and compute the associated least-squares error. Using your answer to the previous question, determine whether the equation* $A\mathbf{x} = \mathbf{b}$ *is consistent.*

**Exercise 2** (10 points)**.** *Fit a polynomial of degree at most* 2 *to the data points* $(0, 27)$*,* $(1, 0)$*,* $(2, 0)$*,* $(3, 0)$ *using least squares.*

**Exercise 3** (5 points)**.** *Consider the following permutation in* $S_7$*:*

$$\pi \quad = \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 2 & 3 & 6 & 5 & 1 & 4 \end{pmatrix}.$$

*Compute the matrix* $P_\pi$*, and compute* $det(P_\pi)$*.*

> ***Hint:*** $det(P_\pi)$ *can be computed in various ways, but it is easiest to use Proposition 7.1.1 from the Lecture Notes.*

**Exercise 4** (5 points)**.** *Consider the following permutation matrix (with entries understood to be in some field $\mathbb{F}$):*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

*Compute the permutation $\pi \in S_6$ such that $A = P_\pi$, and compute det$(A)$.*

> ***Hint:*** *Same hint as for the previous exercise.*

**Problem 1** (20 points)**.** *Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an orthonormal basis of $\mathbb{R}^n$, and set $A = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_{n-1} \end{bmatrix}$. Find the least-squares solution(s) of the matrix-vector equation $A\mathbf{x} = \mathbf{u}_n$. What is the associated least-squares error?*

**Problem 2** (20 points)**.** *Let $Q_1, Q_2 \in \mathbb{R}^{n\times m}$ and $Q \in \mathbb{R}^{m\times m}$. Assume that matrices $Q_1$ and $Q_2$ have orthonormal columns (with respect to the standard scalar product $\cdot$ and the induced norm $||\cdot||$ in $\mathbb{R}^n$),*[1] *and assume furthermore that $Q_1 = Q_2 Q$. Prove that the matrix $Q$ is orthogonal.*

> ***Hint:*** *Consider the product $Q_1^T Q_1 = (Q_2 Q)^T (Q_2 Q)$*

**Problem 3** (30 points)**.** *Consider the vectors*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}$$

*in $\mathbb{R}^4$, and set $C := Span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$.*

*(a) Compute the standard matrix of orthogonal projection onto $C$.*

*(b) Compute an orthonormal basis for $C$ and an orthonormal basis for $C^\perp$.*

*(c) Let $\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$. Compute vectors $\mathbf{y} \in C$ and $\mathbf{z} \in C^\perp$ such that $\mathbf{x} = \mathbf{y} + \mathbf{z}$.*

---

[1] **Remark:** Despite Theorem 6.8.1 from the Lecture Notes, we cannot conclude that matrices $Q_1$ and $Q_2$ are orthogonal. This is because $Q_1$ and $Q_2$ are not necessarily square matrices. By definition, only square matrices can be orthogonal.