Linear Algebra 2:
HW#3

Todor Antić & Irena Penev
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due Monday, March 18, 2024, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Exercise 1 (15 points)**.** *In this exercise, we consider* $\mathbb{R}^5$ *to be equipped with the standard scalar product* $\cdot$ *and the induced norm* $||\cdot||$*. Consider the following linearly independent vectors in* $\mathbb{R}^5$*:*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix}.$$

(You may assume that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *really are linearly independent.) Using the Gram-Schmidt orthogonalization process (either version), find an orthonormal basis of* $U := Span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$*.*

Exercise 2 (15 points)**.** *In this exercise, we assume that* $\mathbb{R}^4$ *is equipped with the standard scalar product* $\cdot$ *and the induced norm* $||\cdot||$*. Consider the following matrix in* $\mathbb{R}^{4\times 5}$*:*

$$A = \begin{bmatrix} 3 & -5 & 1 & 1 & 4 \\ 1 & 1 & 3 & 1 & 2 \\ -1 & 5 & 3 & -2 & -3 \\ 3 & -7 & -1 & 8 & 11 \end{bmatrix}.$$

Compute an orthonormal basis of $Col(A)$*.*

Problem 1 (20 points)**.** *Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $||\cdot||$. Prove that*

$$||\mathbf{x}-\mathbf{y}||^2 + ||\mathbf{x}+\mathbf{y}||^2 \quad = \quad 2||\mathbf{x}||^2 + 2||\mathbf{y}||^2$$

for all $\mathbf{x}, \mathbf{y} \in V$.

Problem 2 (20 points)**.** *The equality from Problem 1 holds for norms induced by scalar products, but it need not hold for other norms.*

*(a) Prove that the equality from Problem 1 does **not** hold for the Manhattan norm $||\cdot||_1$ on $\mathbb{R}^2$, that is, find some two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ such that*

$$||\mathbf{x}-\mathbf{y}||_1^2 + ||\mathbf{x}+\mathbf{y}||_1^2 \quad \neq \quad 2||\mathbf{x}||_1^2 + 2||\mathbf{y}||_1^2.$$

*(b) Prove that the equality from Problem 1 does **not** hold for the Chebyshev distance $||\cdot||_\infty$ on $\mathbb{R}^2$, that is, find some two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ such that*

$$||\mathbf{x}-\mathbf{y}||_\infty^2 + ||\mathbf{x}+\mathbf{y}||_\infty^2 \quad \neq \quad 2||\mathbf{x}||_\infty^2 + 2||\mathbf{y}||_\infty^2.$$

Problem 3 (30 points)**.** *Let U and V be real vector spaces, equipped with scalar products $\langle \cdot, \cdot \rangle_U$ and $\langle \cdot, \cdot \rangle_V$, respectively. Let $f : U \to V$ be a function that satisfies the property that*

$$\langle f(\mathbf{u}), f(\mathbf{v}) \rangle_V \quad = \quad \langle \mathbf{u}, \mathbf{v} \rangle_U$$

for all $\mathbf{u}, \mathbf{v} \in U$. Prove that f is linear and one-to-one.

> ***Hint:*** *For linearity, consider*
>
> - $\Big\langle f(\mathbf{u}+\mathbf{v}) - f(\mathbf{u}) - f(\mathbf{v}), f(\mathbf{u}+\mathbf{v}) - f(\mathbf{u}) - f(\mathbf{v}) \Big\rangle_V$
> - $\Big\langle f(\alpha\mathbf{u}) - \alpha f(\mathbf{u}), f(\alpha\mathbf{u}) - \alpha f(\mathbf{u}) \Big\rangle_V$
>
> *where $\mathbf{u}, \mathbf{v} \in U$ and $\alpha \in \mathbb{R}$.*