Linear Algebra 2:
HW#2

Todor Antić & Irena Penev
Summer 2024

due Monday, March 11, 2024, at 10 am (Prague time)

> Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Exercise 1 (10 points)**.** *Compute the angle* θ *between the vectors*

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$$

in $\mathbb{R}^3$*. (Your final answer may involve an inverse trigonometric function.) Is the angle* θ *acute, right, or obtuse?*

Problem 1 (30 points)**.** *Let* V *be a real vector space, equipped with a scalar product* $\langle \cdot, \cdot \rangle$ *and the norm* $||\cdot||$ *induced by* $\langle \cdot, \cdot \rangle$*. Prove that if vectors* $\mathbf{x}, \mathbf{y} \in V$ *satisfy*

$$||\mathbf{x} + \mathbf{y}||^2 \quad = \quad ||\mathbf{x}||^2 + ||\mathbf{y}||^2,$$

then they are orthogonal.

> ***Remark:*** *So, you are asked to prove the converse of the Pythagorean theorem for the case of **real** vector spaces.*

Problem 2 (30 points)**.** *Consider the standard scalar product* $\cdot$ *on* $\mathbb{C}^2$*, and the induced norm* $||\cdot||$*. Find **non-orthogonal** vectors* $\mathbf{x}, \mathbf{y} \in \mathbb{C}^2$ *that satisfy*

$$||\mathbf{x} + \mathbf{y}||^2 \quad = \quad ||\mathbf{x}||^2 + ||\mathbf{y}||^2.$$

Remark: *This shows that the converse of the Pythagorean theorem need not hold for* ***complex*** *vector spaces.*

Definition. *The* trace *of a square matrix* $A = \left[\ a_{i,j}\ \right]_{n\times n}$ *is defined to be*

$$trace(A) \quad := \quad \textstyle\sum_{i=1}^{n} a_{i,i} \quad = \quad a_{1,1} + a_{2,2} + \cdots + a_{n,n}.$$

In other words, the trace of a square matrix is the sum of its entries on the main diagonal.[1]

Problem 3 (30 points)**.** *Consider the function* $\langle\cdot,\cdot\rangle : \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m} \to \mathbb{R}$ *given by*

$$\langle A, B\rangle \quad = \quad trace\big(A^T B\big)$$

for all $A, B \in \mathbb{R}^{n\times m}$*. Prove that* $\langle\cdot,\cdot\rangle$ *is a scalar product on* $\mathbb{R}^{n\times m}$*.*

[1] For example, $\text{trace}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right) = 1 + 5 + 9 = 15.$