Linear Algebra 2:
HW#1

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due Monday, March 4, 2024, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Remark #1: Feel free to use a calculator (for example: `https://www.dcode.fr/matrix-row-echelon`) for any row reduction that you perform in this HW assignment. (Make sure you tell the calculator what field you are working over.) However, keep in mind that, on the exam, you may be required to solve problems of this type **without** a calculator.

Remark #2: When working with coordinate vectors and matrices of linear functions, make sure you always **specify the bases** that you are working with.

Exercise 1 (15 points). *Let U and V be real vector spaces (i.e. vector spaces over $\mathbb{R}$), let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ be a basis of U, and let $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be a basis of V. Let $f : U \to V$ be the unique linear function that satisfies the following:*

- $f(\mathbf{b}_1) = 5\mathbf{c}_1 - \mathbf{c}_3$;
- $f(\mathbf{b}_2) = -3\mathbf{c}_2$;
- $f(\mathbf{b}_3) = -\mathbf{c}_1 + \mathbf{c}_2$;
- $f(\mathbf{b}_4) = 3\mathbf{c}_2 + \mathbf{c}_3$.

(The existence and uniqueness of f follow from Theorem 4.3.2 of the Lecture Notes.) Compute the matrix ${}_{\mathcal{C}}\big[\ f\ \big]_{\mathcal{B}}$. Then, compute rank$(f)$ and dim$\big(Ker(f)\big)$. Is f one-to-one? Is it onto? Is it an isomorphism? (Make sure you justify your answer.)

Exercise 2 (15 points)**.** *Consider the following sets of vectors (with entries understood to be in $\mathbb{Z}_3$):*

- $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$;
- $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$.

Prove that $\mathcal{B}$ and $\mathcal{C}$ are bases of $\mathbb{Z}_3^3$, and compute the change of basis matrix ${}_{\mathcal{C}}\Big[\ Id_{\mathbb{Z}_3^3}\ \Big]_{\mathcal{B}}$.

Problem 1 (40 points)**.** *Consider the following polynomials with coefficients in $\mathbb{Z}_2$ and matrices with entries in $\mathbb{Z}_2$:*

- $p_1(x) = x^3 + x$;
- $p_2(x) = x^2 + 1$;
- $p_3(x) = x^3+x^2+x+1$;
- $p_4(x) = x^2 + x$;
- $p_5(x) = x^3 + x + 1$;
- $p_6(x) = x^2$;

- $M_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$;
- $M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$;
- $M_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$;
- $M_4 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$;
- $M_5 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$;
- $M_6 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

(a) *Prove that there exists a unique linear function $f : \mathbb{P}^3_{\mathbb{Z}_2} \to \mathbb{Z}_2^{2\times 2}$ satisfying the property that $f\big(p_i(x)\big) = M_i$ for all indices $i \in \{1, \ldots, 6\}$.*

Notation: *As usual, $\mathbb{P}^3_{\mathbb{Z}_2}$ is the vector space (over $\mathbb{Z}_2$) of all* *polynomials with coefficients in* $\mathbb{Z}_2$ *and of degree at most* 3. $\mathbb{Z}_2^{2\times 2}$ *is the vector space (over* $\mathbb{Z}_2$*) of all* 2×2 *matrices with entries in* $\mathbb{Z}_2$.

(b) *Find a formula for the linear function* f *from part (a). Your final answer should be of the following form:*

$$f\big(a_3x^3 + a_2x^2 + a_1x + a_0\big) \quad = \quad \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \qquad \forall a_0, a_1, a_2, a_3 \in \mathbb{Z}_2$$

with the question marks replaced with the appropriate values.

(c) *Prove that the linear function* f *from part (a) is an isomorphism, and find a formula for* f^{-1}. *Your final answer should be of the following form:*

$$f^{-1}\Big(\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}\Big) \quad = \quad \underline{\hspace{3cm}} \qquad \forall a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2} \in \mathbb{Z}_2$$

with the blank filled in with the appropriate polynomial.

Problem 2 (30 points). *Consider the following polynomials with coefficients in* $\mathbb{Z}_3$*:*

- $p_1(x) = x^2 + 2x$;
- $p_2(x) = x^3$;
- $p_3(x) = x + 1$;
- $p_4(x) = x^3 + 2x^2 + 2x + 1$;
- $p_5(x) = 2x^3 + x + 1$;
- $q_1(x) = 2$;
- $q_2(x) = x^3 + 1$;
- $q_3(x) = 2x^3 + x^2 + 2x + 1$;
- $q_4(x) = x^2 + 2x$;
- $q_5(x) = x^3 + x^2 + 2x$.

Prove that there exists a linear function $f : \mathbb{P}^3_{\mathbb{Z}_3} \to \mathbb{P}^3_{\mathbb{Z}_3}$ *satisfying the property that* $f\big(p_i(x)\big) = q_i(x)$ *for all indices* $i \in \{1, \ldots, 5\}$. *How many such linear functions* f *exist?*