Linear Algebra 2

Complex Numbers

Irena Penev

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    2. The Fundamental Theorem of Algebra

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- The real and imaginary part of a complex number z are denoted by $Re(z)$ and $Im(z)$, respectively.
  - For example, we have the following:
    - $Re(2+i) = 2$ and $Im(2+i) = 1$;
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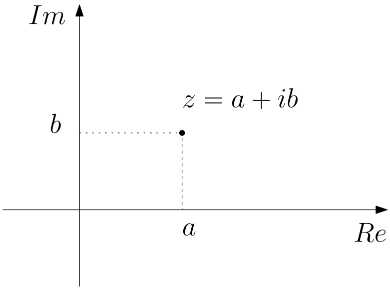
- Note that real numbers are precisely those complex numbers whose imaginary part is zero.
- The set of all complex numbers is denoted by $\mathbb{C}$.

- Complex numbers can be visualized in the "complex plane." This plane has two axes: the *real axis* (denoted by *Re*) and the *imaginary axis* (denoted by *Im*).



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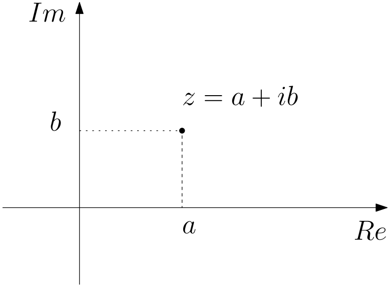

- Note that real numbers are precisely those complex numbers that lie on the real axis.

- We add/subtract complex numbers by adding/subtracting the real and imaginary parts.
  - For example:
    - $(2+3i)+(3-5i) = (2+3)+(3i-5i) = 5-2i$;
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- To multiply complex numbers, we must keep in mind that $i^2 = -1$.
    - For example:

$$\begin{aligned}(2+3i)(3-5i) &= 2\cdot 3 + 2(-5i) + (3i)3 + (3i)(-5i)\\ &= 6 - 10i + 9i - 15\underbrace{i^2}_{=-1}\\ &= 21 - i.\end{aligned}$$

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- Division: later!

Proposition 0.3.1

All the following hold:

(a) addition and multiplication in $\mathbb{C}$ are commutative, that is, for all $z_1, z_2 \in \mathbb{C}$, we have that $z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1$;

(b) addition and multiplication in $\mathbb{C}$ are associative, that is, for all $z_1, z_2, z_3 \in \mathbb{C}$, we have that $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1 z_2) z_3 = z_1 (z_2 z_3)$;

(c) multiplication is distributive over addition in $\mathbb{C}$, that is, for all $z_1, z_2, z_3 \in \mathbb{C}$, we have that $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

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  - $z^0 := 1$;
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  - $z^0 := 1$;
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- So, for all positive integers m, we have the familiar expression

$$z^m = \underbrace{z \dots z}_{m}.$$

Definition

For a complex number $z = a + bi$ (where $a, b \in \mathbb{R}$):

- the *complex conjugate* of z is $\overline{z} := a - bi$;
- the modulus (or absolute value) of z is $|z| := \sqrt{a^2 + b^2}$.

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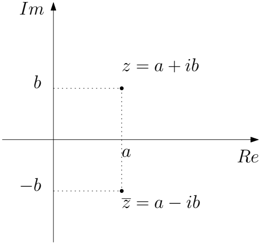

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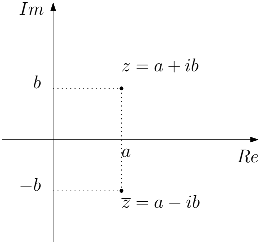

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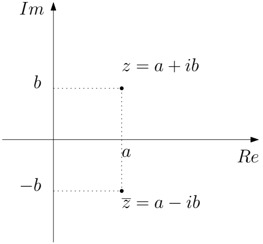

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- Note that $\overline{z} = z$ iff z is in fact a real number, i.e. $Im(z) = 0$.

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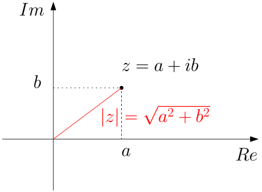

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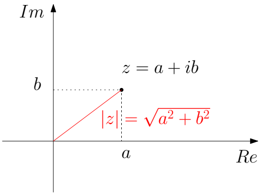

- Note that $|z|$ is a non-negative real number, and moreover, we have that $|z| = 0$ iff $z = 0$.

Proposition 0.3.2

For all complex numbers $z = a + bi$ (with $a, b \in \mathbb{R}$), we have that

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- Note that Proposition 0,3,2, in particular, establishes that multiplying a complex number z by its conjugate produces a real number; that real number is zero iff $z = 0$.
    - This is important for division!

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  - First of all, given a complex number $z = a + bi$ (with $a, b \in \mathbb{R}$) and a real number $r \neq 0$, we have

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    - We do this by multiplying both the numerator and the denominator by $\overline{z_2}$, at which point (by Proposition 0.3.2) the denominator becomes $|z_2|^2$, which is a non-zero real number, and we can divide as above.

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- Let us take a look at an example.

Example 0.3.3

Compute the following quotients:

(a) $\frac{7-6i}{3+2i}$;
(b) $\frac{1}{2-i}$;
(c) $\frac{2-3i}{5}$;
(d) $\frac{4-2i}{2-i}$.

Solution.

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Solution. (a) We multiply both the numerator and the denominator by $\overline{3+2i} = 3-2i$, and we obtain

$$\frac{7-6i}{3+2i} = \frac{(7-6i)(3-2i)}{(3+2i)(3-2i)} = \frac{9-32i}{9+4} = \frac{9}{13} - \frac{32}{13}i.$$

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(b) We multiply both the numerator and the denominator by $\overline{2-i} = 2+i$, and we obtain

$$\frac{1}{2-i} = \frac{2+i}{(2-i)(2+i)} = \frac{2+i}{4+1} = \frac{2}{5} + \frac{1}{5}i.$$

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Solution (continued). (c) The denominator is a real number, and so we have

$$\frac{2-3i}{5} = \frac{2}{5} - \frac{3}{5}i.$$

(d) We could multiply both the numerator and the denominator by $\overline{2-i} = 2+i$. However, in this particular case, it is easier to compute as follows:

$$\frac{4-2i}{2-i} = \frac{2(2-i)}{2-i} \stackrel{(*)}{=} 2,$$

where (*) was obtained by canceling out the common factor $2-i$ in the numerator and the denominator. $\square$

Proposition 0.3.4

For all $z_1, z_2 \in \mathbb{C}$, the following hold:

(a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$;

(b) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$;

(c) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$;

(d) if $z_2 \neq 0$, then $\overline{z_1/z_2} = \overline{z_1}/\overline{z_2}$.

Moreover, for all $z \in \mathbb{C}$ and non-negative integers m, we have that

(e) $\overline{z^m} = (\overline{z})^m$.

Proposition 0.3.5

For all $z_1, z_2 \in \mathbb{C}$, the following hold:

(a) $|z_1 z_2| = |z_1||z_2|$;

(b) if $z_2 \neq 0$, then $|z_1/z_2| = |z_1|/|z_2|$.

Moreover, for all $z \in \mathbb{C}$, the following hold:

(c) $|-z| = |z|$;

(d) for all non-negative integers m, we have $|z^m| = |z|^m$.

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- In the particular case of $p(x) = x^2 - 2x + 2$, the roots could have been found via the familiar quadratic equation.
- There exist formulas for finding the complex roots of all third and fourth degree polynomials with complex coefficients, but no such formula exists for polynomials of degree five or more (although in some special cases, we may be able to use various tricks to find the roots of these higher-degree polynomials).

- Nevertheless, we do have the following **existence** result.
    - A *constant* polynomial is a polynomial of the form $p(x) = c$, where c is a fixed constant/number.

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  - So, the Fundamental Theorem of Algebra, in particular, implies that every non-constant polynomial with real coefficients has a complex root (which may or may not be a real number).
  - For instance, the polynomial $p(x) = x^2 + 1$ is a non-constant polynomial with real (in fact, rational) coefficients, but it has no real roots. It does, of course, have two complex roots, namely i and $-i$.

The Fundamental Theorem of Algebra

Any non-constant polynomial with complex coefficients has a complex root.

- We omit the proof of the Fundamental Theorem of Algebra.
- There are no known elementary proofs of this theorem: all the known proofs of the Fundamental Theorem of Algebra rely on advanced mathematics, such as complex analysis or topology.

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  - Here, a is the leading coefficient of $p(x)$, i.e. the coefficient in front of x^n. Complex numbers $\alpha_1, \dots, \alpha_\ell$ are the roots of $p(x)$ with *multiplicities* $n_1, \dots, n_\ell$, respectively.

- The Fundamental Theorem of Algebra implies that any polynomial $p(x)$ with complex coefficients and of degree $n \geq 1$ can be factored into n linear factors.
- More precisely, for such a polynomial $p(x)$, there exist complex numbers $a, \alpha_1, \dots, \alpha_\ell$ such that $a \neq 0$ and such that $\alpha_1, \dots, \alpha_\ell$ are pairwise distinct, and positive integers $n_1, \dots, n_\ell$ satisfying $n_1 + \dots + n_\ell = n$, such that

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    - If we think of each α_i as being a root "n_i times" (due to its multiplicity), then we see that the n-th degree polynomial $p(x)$ has exactly n complex roots.
    - This is often summarized as follows: "every n-th degree polynomial (with $n \geq 1$) with complex coefficients has exactly n complex roots, when multiplicities are taken into account."

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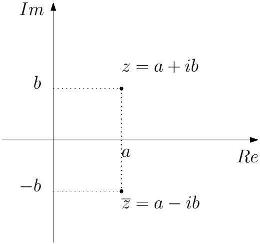
- In fact, not only is no such formula known, but using Galois theory, one can show that no such formula can exist for polynomials of degree at least five.
  - Once again, we may be able to use various tricks to compute the roots of some special high-degree polynomials. However, none of these tricks will work in the general case.

- Recall that, geometrically, the complex conjugate of a complex number *z* is obtained by reflecting *z* about the *Re* axis in the complex plane.



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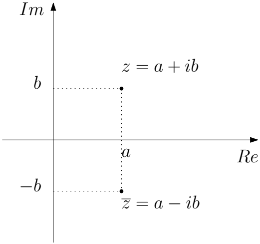

Theorem 0.3.6

Let $p(x)$ be any polynomial with **real** coefficients, and let $z \in \mathbb{C}$. Then z is a root of $p(x)$ iff $\overline{z}$ is a root of $p(x)$.

- First a remark, then a proof.

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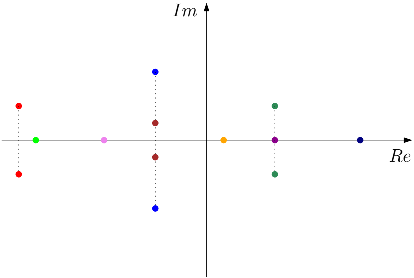
- **Remark:** Note that Theorem 0.3.6 implies that the complex roots of a non-constant polynomial are symmetric about the *Re* axis in the complex plane.
    - Some (or perhaps all) of those roots may lie on the *Re* axis, i.e. they may be real numbers.



Theorem 0.3.6

Let $p(x)$ be any polynomial with **real** coefficients, and let $z \in \mathbb{C}$. Then z is a root of $p(x)$ iff $\overline{z}$ is a root of $p(x)$.

Proof. Set $p(x) = a_n x^n + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_n \in \mathbb{R}$. Then we have the following sequence of equivalences:

$$
\begin{aligned}
p(z) = 0 \quad &\Longleftrightarrow \quad \overline{p(z)} = \overline{0} \\
&\Longleftrightarrow \quad \overline{a_n z^n + \cdots + a_1 z + a_0} = \overline{0} \\
&\overset{(*)}{\Longleftrightarrow} \quad \overline{a_n}(\overline{z})^n + \cdots + \overline{a_1}(\overline{z}) + \overline{a_0} = \overline{0} \\
&\overset{(**)}{\Longleftrightarrow} \quad a_n(\overline{z})^n + \cdots + a_1\overline{z} + a_0 = 0 \\
&\Longleftrightarrow \quad p(\overline{z}) = 0,
\end{aligned}
$$

where (*) follows from Proposition 0.3.4, and (**) follows from the fact that $a_0, a_1, \ldots, a_n$ and 0 are real numbers. $\square$