

Linear Algebra 2: Tutorial 13

Irena Penev

Summer 2023

Problem 1 of HW#7. Let $\langle \cdot, \cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = 2x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2 - x_2y_3 - x_3y_2 + 2x_3y_3$$

for all $\mathbf{x} = [x_1 \ x_2 \ x_3]^T$ and $\mathbf{y} = [y_1 \ y_2 \ y_3]^T$ in \mathbb{R}^3 . Determine whether $\langle \cdot, \cdot \rangle$ is a scalar product in \mathbb{R}^3 . Make sure you justify your answer.

Exercise 1.

(a) Consider the quadratic form $q : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$q(\mathbf{x}) = x_1^2 + x_1x_2 + x_2^2 \quad \text{for all } \mathbf{x} = [x_1 \ x_2]^T \text{ in } \mathbb{R}^2.$$

Find the symmetric matrix of the quadratic form q with respect to the standard basis. (So, you need to find the unique symmetric matrix $A \in \mathbb{R}^{2 \times 2}$ such that $q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$.)

(b) Same question as in part (a), but replace \mathbb{R} by \mathbb{Z}_3 throughout.

Exercise 2. Give an example of a quadratic form $q : \mathbb{Z}_2^2 \times \mathbb{Z}_2^2 \rightarrow \mathbb{Z}_2$ for which there does **not** exist a symmetric matrix $A \in \mathbb{Z}_2^{2 \times 2}$ such that

$$q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{Z}_2^2.$$

Prove that your example is correct.

Remark: Here, the point is that $\text{char}(\mathbb{Z}_2) = 2$. For fields of characteristic other than 2, such a symmetric matrix A always exists (by Theorem 3.1 of Lecture Notes 22).

Exercise 3. Let \mathbb{F} be a field, and let $A = [a_{i,j}]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. Consider the quadratic form $q : \mathbb{F}^n \times \mathbb{F}^n \rightarrow \mathbb{F}$ given by

$$q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{F}^n.$$

Compute $q(\mathbf{e}_1)$.

Exercise 4. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix, and suppose that $\lambda \in \mathbb{R}$ is an eigenvalue of A , and that $\mathbf{v} \in \mathbb{R}^n$ is an associated unit eigenvector.¹ Consider the quadratic form $q: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Compute $q(\mathbf{v})$.

Exercise 5. Prove or disprove the following statement:

For every symmetric matrix $A \in \mathbb{R}^{n \times n}$, if there exists a vector $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\mathbf{x}^T A \mathbf{x} = 0$, then A is non-invertible.

Exercise 6. Let \mathbb{F} be a field, and let $A \in \mathbb{F}^{n \times m}$ and $\mathbf{b} \in \mathbb{F}^m$. Without using the results of section 4 of Lecture Notes 23, show that the solution set S of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is either empty or is an affine subspace of \mathbb{F}^m . (Remark: $S \neq \emptyset$ if and only if $A\mathbf{x} = \mathbf{b}$ is consistent.) If $S \neq \emptyset$, then find a suitable vector $\mathbf{a} \in \mathbb{F}^m$ and a suitable linear subspace U of \mathbb{F}^m such that $S = \mathbf{a} + U$.

Definition. Let n be a non-negative integer, and let M be an n -dimensional affine subspace of a vector space V over a field \mathbb{F} . An affine frame of M is an ordered $(n+1)$ -tuple $(\mathbf{a}, \mathbf{u}_1, \dots, \mathbf{u}_n)$ of vectors of V such that M can be written in the form $M = \mathbf{a} + U$, where U is a linear subspace of V , and $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is a basis of U .

Exercise 7. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

with entries understood to be in \mathbb{Z}_2 . Is the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ consistent? If so, find an affine frame of its solution set.

¹“Unit” means that $\|\mathbf{v}\| = 1$.