Linear Algebra 2: Tutorial 12

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Summer 2023

Exercise 1. *For each of the bilinear forms on $\mathbb{R}^n$ below, find the matrix of the bilinear form with respect to the standard basis, determine if the bilinear form is symmetric, and determine if it is a scalar product in $\mathbb{R}^n$.*

(a) $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, *given by* $f(\mathbf{x}, \mathbf{y}) = x_1y_1 - x_1y_2 + x_2y_1 - x_2y_2$ *for all* $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ *and* $\mathbf{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$ *in* $\mathbb{R}^2$;

(b) $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, *given by* $f(\mathbf{x}, \mathbf{y}) = 2x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$ *for all* $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ *and* $\mathbf{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$ *in* $\mathbb{R}^2$;

(c) $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, *given by* $f(\mathbf{x}, \mathbf{y}) = x_1y_1 + 2x_1y_2 + 3x_1y_3 + 2x_2y_1 + x_2y_2 - x_2y_3 + 3x_3y_1 - x_3y_2 + 4x_3y_3$ *for all* $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ *and* $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T$ *in* $\mathbb{R}^3$.

Exercise 2. *Consider the following matrices (with indices understood to be in $\mathbb{C}$).*

(a) $A = \begin{bmatrix} 4 & 9 \\ -1 & -2 \end{bmatrix}$;

(b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$;

(c) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$;

(d) $A = \begin{bmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

For each of the matrices above:

1. *find the characteristic polynomial of the matrix A, all the eigenvalues (along with their algebraic and geometric multiplicities), as well as a basis for each of the eigenspaces,*
2. *either diagonalize A or explain why it is not diagonalizable,*
3. *if A is diagonalizable, then find a formula for A^m (for a non-negative integer m), and determine whether the formula is also correct for negative integers m.*