Linear Algebra 2: Tutorial 7

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Exercise 1. *Let a, b, c be positive real numbers. Compute the volume of the solid enclosed by the ellipsoid* $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1 \right\}$.

Exercise 2. *Prove or disprove the following statement:*

> *For all matrices* $A = \begin{bmatrix} \mathbf{a}_1 & \dots \mathbf{a}_n \end{bmatrix}$ *in* $\mathbb{R}^{n \times n}$*, we have that* $|det(A)| \leq \prod_{i=1}^{n} ||\mathbf{a}_i||$.
>
> ***Hint:*** *Volume.*

Exercise 3. *Let n be a positive integer. What is the maximum possible value of $det(A)$ if A is a matrix in $\mathbb{R}^{(2n)\times(2n)}$, all of whose entries are 1, 0, or -1?*

> ***Hint:*** *First think about $n = 1$, then about $n = 2$, and then generalize.*

Exercise 4. *Prove of disprove the following statement:*

> *For all invertible matrices $A \in \mathbb{R}^{2\times 2}$ and $\mathbf{v} \in \mathbb{R}^2$, we have that* $||A\mathbf{v}|| \leq det(A)||\mathbf{v}||$.

Exercise 5. *Either construct a matrix $A \in \mathbb{R}^{3\times 3}$ such that $A^2 = -I_3$, or prove that no such matrix exists.*

Exercise 6. *Either construct invertible matrices $P, A \in \mathbb{R}^{3\times 3}$ such that $P^T AP = -A$, or prove that no such matrices exist.*

Exercise 7. *Prove or disprove the each of the following statements.*

(a) For all $A, B \in \mathbb{R}^{2\times 2}$*,* $det(A + B) \neq det(A) + det(B)$*.*

(b) For all $A \in \mathbb{R}^2$*, there exists some* $B \in \mathbb{R}^{2\times 2}$ *such that* $det(A + B) \neq det(A) + det(B)$*.*

Exercise 8. *For which (if any) real values of* k *is the matrix*

$$A = \begin{bmatrix} k^2 & 1 & 4 \\ k & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

invertible?

Exercise 9. *Show that the area of the triangle with vertices* (x_1, y_1)*,* (x_2, y_2)*, and* (x_3, y_3) *in* $\mathbb{R}^2$ *is equal to*

$$\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|.$$

Exercise 10. *For invertible matrices* $A, B \in \mathbb{R}^{n\times n}$*, what is the relationship between* $adj(A)$*,* $adj(B)$*, and* $adj(AB)$*?*