Linear Algebra 2: Tutorial 6

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Summer 2023

Definition 1. *A* scalar product *(also called* inner product*) in a vector space V over the field $\mathbb{R}$ is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies the following axioms:*

r.1. for all $\mathbf{x} \in V$, $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, *and equality holds if and only if* $\mathbf{x} = \mathbf{0}$;

r.2. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$;

r.3. for all $\mathbf{x}, \mathbf{y} \in V$ *and* $\alpha \in \mathbb{R}$, $\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha\langle \mathbf{x}, \mathbf{y} \rangle$;

r.4. for all $\mathbf{x}, \mathbf{y} \in V$, $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.

Definition 2. *The* trace *of a square matrix* $A = [a_{i,j}]_{n \times n}$ *is* $trace(A) = \sum_{i=1}^{n} a_{i,i}$.[1]

Problem 3 from HW#2. *Consider the function* $\langle \cdot, \cdot \rangle : \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times m} \to \mathbb{R}$ *given by*

$$\langle A, B \rangle \quad = \quad trace(A^T B)$$

for all $A, B \in \mathbb{R}^{n \times m}$. *Prove that* $\langle \cdot, \cdot \rangle$ *is a scalar product on* $\mathbb{R}^{n \times m}$.

Exercise 1. *Let* $A \in \mathbb{R}^{n \times n}$. *If* $det(A) = 3$, *then what is* $det(A^T A)$?

Exercise 2. *Let* $A \in \mathbb{R}^{n \times n}$. *What can you say about the sign of* $det(A^T A)$?

[1] In other words, the *trace* of a square matrix is the sum of its entries on the main diagonal. For example, $\text{trace}\Big(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\Big) = 1 + 5 + 9 = 15$.

Definition 3. *Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$.*

- *A is* symmetric *if $A^T = A$.*
- *A is* skew-symmetric *of $A^T = -A$.*

Exercise 3. *Let $\mathbb{F}$ be a field of characteristic other than 2, i.e. assume that in the field $\mathbb{F}$, $1 + 1 \neq 0$.*

(a) Show that if n is odd positive integer, then any skew-symmetric matrix $A \in \mathbb{F}^{n\times n}$ is non-invertible. (Use determinants.)

(b) For each even positive integer n, construct an invertible skew-symmetric matrix in $\mathbb{F}^{n\times n}$.

Exercise 4. *For real numbers $a_0, a_1, \ldots, a_n$, we define the $(n+1) \times (n+1)$ matrix*

$$V(a_0, a_1, \ldots, a_n) \quad = \quad \begin{bmatrix} 1 & 1 & \ldots & 1 \\ a_0 & a_1 & \ldots & a_n \\ a_0^2 & a_1^2 & \ldots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_0^n & a_1^n & \ldots & a_n^n \end{bmatrix},$$

called the Vandermonde matrix*. The goal of this exercise is to prove that*

$$det\Big(V(a_0, a_1, \ldots, a_n)\Big) \quad = \quad \prod_{i>j} (a_i - a_j).$$

(a) Verify the above formula for the case when $n = 1$.

(b) Now fix a positive integer n, and assume inductively that the formula is correct for $(n+1) \times (n+1)$ Vandermonde matrices.[2] Fix real numbers $a_0, a_1, \ldots, a_n, a_{n+1}$.

(b.1) What happens if some two of $a_0, a_1, \ldots, a_n, a_{n+1}$ are equal?

(b.2) From now on, assume that $a_0, a_1, \ldots, a_n, a_{n+1}$ are pairwise distinct, and set

$$f(t) \quad := \quad \begin{vmatrix} 1 & 1 & \ldots & 1 & 1 \\ a_0 & a_1 & \ldots & a_n & t \\ a_0^2 & a_1^2 & \ldots & a_n^2 & t^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_0^n & a_1^n & \ldots & a_n^n & t^n \\ a_0^{n+1} & a_1^{n+1} & \ldots & a_n^{n+1} & t^{n+1} \end{vmatrix}.$$

[2]This means that for all $a_0, a_1, \ldots, a_n \in \mathbb{R}$, we have that $\det\Big(V(a_0, a_1, \ldots, a_n)\Big) = \prod_{i>j} (a_i - a_j)$.

Explain why $f(t)$ is an $(n+1)^{th}$ degree polynomial. What is the leading coefficient (i.e. the coefficient in front of t^{n+1}) of $f(t)$?

(b.3) *What are the roots of the polynomial $f(t)$? Using your answer, factor the polynomial $f(t)$.*

(b.4) *Prove that* $det\Big(V(a_0, a_1, \ldots, a_n, a_{n+1})\Big) = \prod_{i>j}(a_i - a_j)$.

Exercise 5. *Compute the determinant*

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \\ 1 & 16 & 81 & 256 & 625 \end{vmatrix}.$$