Linear Algebra 2: Tutorial 4

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Problem 3 from HW#1.. *As usual, $\mathbb{Z}_2^{2\times 2}$ is the vector space (over $\mathbb{Z}_2$) of all 2×2 matrices with entries in $\mathbb{Z}_2$, and $\mathbb{P}_{\mathbb{Z}_2}$ is the vector space (over $\mathbb{Z}_2$) of all polynomials with coefficients in $\mathbb{Z}_2$.*

(a) [20 points] Prove that there exists a unique linear transformation $f : \mathbb{Z}_2^{2\times 2} \to \mathbb{P}_{\mathbb{Z}_2}$ that satisfies all the following:

- $f\Big(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\Big) = x^4 + x^3 + x^2 + x + 1;$
- $f\Big(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\Big) = x^3 + x^2 + x + 1;$
- $f\Big(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\Big) = x^2 + 1;$
- $f\Big(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\Big) = x^4 + x^2 + 1;$
- $f\Big(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\Big) = x^5 + x^4 + x^3 + x + 1;$
- $f\Big(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\Big) = x^5 + x^3 + x^2 + x.$

(b) [15 points] Find rank(f), and determine whether f is one-to-one, where f is the linear transformation from part (a).

(c) [15 points] Find the formula for the linear transformation $f : \mathbb{Z}_2^{2\times 2} \to \mathbb{P}_{\mathbb{Z}_2}$ from part (a), that is, fill in the blank in the following:

$$f\Big(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\Big) \quad = \quad \underline{\qquad\qquad\qquad\qquad} \qquad \forall a, b, c, d \in \mathbb{Z}_2.$$

Exercise 1. *Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Find the least-squares solution(s) of the matrix-vector equation $A\mathbf{x} = \mathbf{u}_n$, where $A = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_{n-1} \end{bmatrix}$.*

The Interpolation Theorem. *For all pairwise distinct* $x_0, x_1, \dots, x_n \in \mathbb{R}$ *and all (not necessarily distinct)* $y_0, y_1, \dots, y_n$*, there exists a unique polynomial* $p(x)$ *of degree at most* n *such that* $p(x_i) = y_i$ *for all* $i = 0, 1, \dots, n$*.*

> ***Remark:*** *This theorem essentially states that if we are given* $n+1$ *data points* $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ *(with* $x_0, x_1, \dots, x_n$ *pairwise distinct), then there is a unique polynomial of degree at most* n *whose graph passes through those data points.*

Proof. Omitted. □

Exercise 2. *Find a polynomial of degree at most* 3 *that passes through the points* $(1, 3)$*,* $(-1, 13)$*,* $(2, 1)$*,* $(-2, 33)$*.*

Exercise 3. *Fit a linear function of the the form* $f(t) = c_0 + c_1 t$ *to the data points* $(0, 0)$*,* $(0, 1)$*,* $(1, 1)$ *using least squares.*

Exercise 4. *The following table gives world population in 10-year intervals, starting with 1950 and ending with 2000.*

year	*population (in* 10^9*)*
1950	*2.519*
1960	*2.982*
1970	*3.692*
1980	*4.435*
1990	*5.263*
2000	*6.070*

We would like to use the least-squares method to find the linear function that best fits these data. Set up the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ *to which we need to apply the least-squares method.*

Exercise 5. *Using the least-squares method, fit a quadratic function to the four data points* $(a_1, b_1) = (-1, 8)$*,* $(a_2, b_2) = (0, 8)$*,* $(a_3, b_3) = (1, 4)$*, and* $(a_4, b_4) = (2, 16)$*.*

Exercise 6. *Explain how one would a trigonometric function of the form* $f(t) = c_0 + c_1 \sin t + c_2 \cos t$ *that best fits the data points* $(0, 0)$*,* $(1, 1)$*,* $(2, 2)$*,* $(3, 3)$*.*

> ***Remark:*** *So, you should set up the matrix-vector equation* $A\mathbf{c} = \mathbf{b}$ *(where* A *and* $\mathbf{b}$ *are known and* $\mathbf{c}$ *is the unknown) to which the least-squares method should be applied. Don't compute the actual least-squares solution.*