Linear Algebra 2: Tutorial 3

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Exercise 5 from Tutorial 1.

(a) *Determine if there exists a linear transformation* $f : \mathbb{Z}_2^{2\times 3} \to \mathbb{Z}_2^3$ *satisfying the following properties:*

- $f(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$;
- $f(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$;
- $f(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$.

(b) *Is the linear transformation* f *from part (a) unique?*

(c) *Can a linear transformation* f *satisfying the properties from part (a) be one-to-one? Can it be onto?*

(d) *Find a formula for some linear transformation* f *satisfying the properties from part (a). Can you find more than one correct answer? Can you come up with examples of different rank?*

Theorem 2.3 from Lecture Notes 13. *Let* $A \in \mathbb{R}^{n\times m}$ *be a matrix of rank* m*. Then the matrix* $A(A^TA)^{-1}A^T$ *is the standard matrix of the orthogonal projection onto Col(A), that is, for all* $\mathbf{x} \in \mathbb{R}^n$*, the orthogonal projection of* $\mathbf{x}$ *onto* $C :=$ *Col(A) is given by*

$$\mathbf{x}_C \quad = \quad A(A^TA)^{-1}A^T\mathbf{x}.$$

Exercise 1. *In what follows, we assume that* $\mathbb{R}^3$ *is equipped with the standard scalar product* $\cdot$*. Consider the following vectors in* $\mathbb{R}^3$*:*

$$\mathbf{v}_1 \ = \ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 \ = \ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_1 \ = \ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{x} \ = \ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Set $C := Span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$*.*

(a) *Compute the standard matrix P of the orthogonal projection $Proj_C$: $\mathbb{R}^3 \to \mathbb{R}^3$ onto C.*

> ***Warning:*** *The matrix $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ does* ***not*** *have rank 3. So, you cannot use Theorem 2.3 from Lecture Notes 13 directly.*

(b) *Using the matrix P from part (a), compute the vector $\mathbf{x}_C$ (the projection of $\mathbf{x}$ onto C). Does $\mathbf{x}$ belong to C?*

(c) *Compute a basis for $C^\perp$. Do this in two different ways: first using Gram-Schmidt orthogonalization, and then using the fact (proven in class) that $Row(A)^\perp = Nul(A)$ for all matrices A.*

Exercise 2.

(a) *Let V be a vector space over $\mathbb{R}$, equipped with the scalar product $\langle \cdot, \cdot \rangle$, and let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an ~~orthogonal~~ orthonormal basis of V. Let $\cdot$ be the standard scalar product in $\mathbb{R}^n$. Prove that for all $\mathbf{x}, \mathbf{y} \in V$, we have that*

$$\langle \mathbf{x}, \mathbf{y} \rangle \quad = \quad [\mathbf{x}]_\mathcal{B} \cdot [\mathbf{y}]_\mathcal{B}.$$

> ***Hint:*** *The first thing you should do is find a formula for $[\mathbf{x}]_\mathcal{B}$ and $[\mathbf{y}]_\mathcal{B}$ using the ~~orthogonal~~ orthonormal basis $\mathcal{B}$ and the scalar product $\langle \cdot, \cdot \rangle$ in V.*

(b) *Same question as in part (a), only with $\mathbb{C}$ instead of $\mathbb{R}$.*