Linear Algebra 2: Tutorial 2

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Summer 2023

Exercise 4 from Tutorial 1.

(a) Prove that there exists a linear transformation $f : \mathbb{Z}_3^4 \rightarrow \mathbb{Z}_3^2$ *such that all the following hold:*

- $f\Big(\begin{bmatrix} 1 & 1 & 2 & 2 \end{bmatrix}^T\Big) = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$;
- $f\Big(\begin{bmatrix} 2 & 2 & 1 & 1 \end{bmatrix}^T\Big) = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$;
- $f\Big(\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^T\Big) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$;
- $f\Big(\begin{bmatrix} 0 & 2 & 2 & 1 \end{bmatrix}^T\Big) = \begin{bmatrix} 2 & 0 \end{bmatrix}^T$.

(b) Is the function f *from part (a) unique?*

(c) Find a formula for some function f *satisfying the properties from part (a). (It is possible that there is more than one correct answer.)*

Exercise 5 from Tutorial 1.

(a) Determine if there exists a linear transformation $f : \mathbb{Z}_2^{2\times 3} \rightarrow \mathbb{Z}_2^3$ *satisfying the following properties:*

- $f(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$;
- $f(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$;
- $f(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$.

(b) Is the linear transformation f *from part (a) unique?*

(c) Can a linear transformation f *satisfying the properties from part (a) be one-to-one? Can it be onto?*

(d) *Find a formula for some linear transformation f satisfying the properties from part (a). Can you find more than one correct answer? Can you come up with examples of different rank?*

Exercise 1. *Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be non-zero vectors in V that are scalar multiples of each other. Prove that for all $\mathbf{x} \in V$, we have that*

$$proj_{\mathbf{u}_1}(\mathbf{x}) \quad = \quad proj_{\mathbf{u}_2}(\mathbf{x}).$$

Exercise 2. *Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the norm $|| \cdot ||$ induced by $\langle \cdot, \cdot \rangle$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a (not necessarily linearly independent) set of vectors in V. Explain how one can obtain an orthogonal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ for $Span(\mathbf{v}_1, \ldots, \mathbf{v}_k)$.*

> ***Remark:*** *To apply Gram-Schmidt, you need to start with a linearly independent set of vectors. Here, you need to adapt Gram-Schmidt so that it works for sets of vectors that are not necessarily linearly independent.*

Exercise 3. *In this exercise, we assume that $\mathbb{R}^4$ is equipped with the standard scalar product $\bullet$, and we consider the matrix*

$$A \quad = \quad \begin{bmatrix} 3 & -5 & 1 & 1 & 4 \\ 1 & 1 & 3 & 1 & 2 \\ -1 & 5 & 3 & -2 & -3 \\ 3 & -7 & -1 & 8 & 11 \end{bmatrix}$$

in $\mathbb{R}^{4\times 5}$. Using the procedure that you described in Exercise 2, find an orthogonal basis for $Col(A)$.