Linear Algebra 2: Tutorial 1

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Summer 2023

Exercise 1. *Let U be a vector space over $\mathbb{Z}_2$, and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for U. Using coordinate vectors, determine whether $\mathcal{C} = \{\mathbf{b}_1 + \mathbf{b}_2, \mathbf{b}_2 + \mathbf{b}_3, \mathbf{b}_3 + \mathbf{b}_1\}$ is a basis for U, and if so, find the transition (i.e. change of basis) matrices ${}_{\mathcal{C}}[Id_U]_{\mathcal{B}}$ and ${}_{\mathcal{B}}[Id_U]_{\mathcal{C}}$.*

Exercise 2. *Let U and V be vector spaces over $\mathbb{Z}_3$, let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for U, and let $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4\}$ be a basis for V. Let $f : U \to V$ be the unique linear transformation such that*

- $f(\mathbf{b}_1) = \mathbf{c}_1 + \mathbf{c}_2$;
- $f(\mathbf{b}_2) = \mathbf{c}_2 + 2\mathbf{c}_3 + 2\mathbf{c}_4$;
- $f(\mathbf{b}_3) = \mathbf{c}_2 + \mathbf{c}_3$.

Find the matrix ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ and compute rank(f). If f one-to-one? Is f onto? Is f an isomorphism?

Exercise 3. *In the following, there may be more than one correct answer for the bases $\mathcal{B}$ and $\mathcal{B}^*$. However, try to find a general (rather than haphazard) method for solving problems of this type.*

(a) Find a basis $\mathcal{B}$ of

$$Span\Big(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\Big)$$

(with entries understood to be in $\mathbb{Z}_3$). Then, find a basis $\mathcal{B}^*$ of $\mathbb{Z}_3^3$ that extends $\mathcal{B}$.[1]

[1] "$\mathcal{B}^*$ extends $\mathcal{B}$" means that $\mathcal{B} \subseteq \mathcal{B}^*$.

(b) *Find a basis* $\mathcal{B}$ *of*

$$Span(x^2+1,\ x^3+x^2+1,\ x+1,\ x^3+x+1,\ x^3+x^2+x)$$

(with coefficients understood to be in $\mathbb{Z}_2$*). Then, find a basis* $\mathcal{B}^*$ *of* $\mathbb{P}^3_{\mathbb{Z}_2}$ *that extends* $\mathcal{B}$*.*

(c) *Find a basis* $\mathcal{B}$ *of*

$$Span\Big(\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}\Big)$$

(with coefficients understood to be in $\mathbb{Z}_3$*). Then, find a basis* $\mathcal{B}^*$ *of* $\mathbb{Z}_3^{2\times 3}$ *that extends* $\mathcal{B}$*.*

Exercise 4.

(a) *Prove that there exists a linear transformation* $f : \mathbb{Z}_3^4 \to \mathbb{Z}_3^2$ *such that all the following hold:*

- $f\Big(\begin{bmatrix} 1 & 1 & 2 & 2 \end{bmatrix}^T\Big) = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$;
- $f\Big(\begin{bmatrix} 2 & 2 & 1 & 1 \end{bmatrix}^T\Big) = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$;
- $f\Big(\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^T\Big) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$;
- $f\Big(\begin{bmatrix} 0 & 2 & 2 & 1 \end{bmatrix}^T\Big) = \begin{bmatrix} 2 & 0 \end{bmatrix}^T$.

(b) *Is the function* f *from part (a) unique?*

(c) *Find a formula for some function* f *satisfying the properties from part (a). (It is possible that there is more than one correct answer.)*

Exercise 5.

(a) *Determine if there exists a linear transformation* $f : \mathbb{Z}_2^{2\times 3} \to \mathbb{Z}_2^3$ *satisfying the following properties:*

- $f(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$;
- $f(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$;
- $f(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$.

(b) Is the linear transformation f from part (a) unique?

(c) Can a linear transformation f satisfying the properties from part (a) be one-to-one? Can it be onto?

(d) Find a formula for some linear transformation f satisfying the properties from part (a). Can you find more than one correct answer? Can you come up with examples of different rank?