Linear Algebra 2

Lecture #23
Affine subspaces

Irena Penev

May 11, 2023

- **Terminology:**

- **Terminology:**
  - So far, we have typically referred to vector/linear subspaces of a vector space simply as "subspaces."

- **Terminology:**
  - So far, we have typically referred to vector/linear subspaces of a vector space simply as "subspaces."
  - In this lecture, we will study a generalization of linear subspaces, called "affine subspaces."

- **Terminology:**
    - So far, we have typically referred to vector/linear subspaces of a vector space simply as "subspaces."
    - In this lecture, we will study a generalization of linear subspaces, called "affine subspaces."
    - To avoid any confusion, in this lecture, we will not use the term "subspace" and will instead always write either "linear subspace" or "affine subspace."

- This lecture has four parts:

- This lecture has four parts:
  1. affine subspaces;

- This lecture has four parts:
  1. affine subspaces;
  2. affine combinations and affine hulls;

- This lecture has four parts:
  1. affine subspaces;
  2. affine combinations and affine hulls;
  3. affine frames and affine bases;

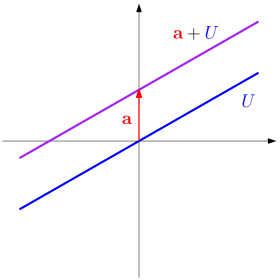
- This lecture has four parts:
  1. affine subspaces;
  2. affine combinations and affine hulls;
  3. affine frames and affine bases;
  4. affine transofrmations.

1. Affine subspaces

1. Affine subspaces

Definition

An *affine subspace* of a vector space V over a field $\mathbb{F}$ is any set of the form $\mathbf{a} + U := \{\mathbf{a} + \mathbf{u} \mid \mathbf{u} \in U\}$ where $\mathbf{a}$ is a vector in V and U is a linear subspace of V.



- Thus, an affine subspace of V is obtained by shifting a linear subspace U of V by some vector $\mathbf{a}$.

Definition

An *affine subspace* of a vector space V over a field $\mathbb{F}$ is any set of the form $\mathbf{a} + U := \{\mathbf{a} + \mathbf{u} \mid \mathbf{u} \in U\}$ where $\mathbf{a}$ is a vector in V and U is a linear subspace of V.

- **Remarks:**

Definition

An *affine subspace* of a vector space V over a field $\mathbb{F}$ is any set of the form $\mathbf{a} + U := \{\mathbf{a} + \mathbf{u} \mid \mathbf{u} \in U\}$ where $\mathbf{a}$ is a vector in V and U is a linear subspace of V.

- **Remarks:**
  1. For a vector space V over a field $\mathbb{F}$:

Definition

An *affine subspace* of a vector space V over a field $\mathbb{F}$ is any set of the form $\mathbf{a} + U := \{\mathbf{a} + \mathbf{u} \mid \mathbf{u} \in U\}$ where $\mathbf{a}$ is a vector in V and U is a linear subspace of V.

- **Remarks:**
  1. For a vector space V over a field $\mathbb{F}$:
     - every linear subspace U of V is also an affine subspace of V, since $U = \mathbf{0} + U$;

Definition

An *affine subspace* of a vector space V over a field $\mathbb{F}$ is any set of the form $\mathbf{a} + U := \{\mathbf{a} + \mathbf{u} \mid \mathbf{u} \in U\}$ where $\mathbf{a}$ is a vector in V and U is a linear subspace of V.

- **Remarks:**
  1. For a vector space V over a field $\mathbb{F}$:
     - every linear subspace U of V is also an affine subspace of V, since $U = \mathbf{0} + U$;
     - V is an affine subspace of itself (because V is a linear subspace of itself);

Definition

An *affine subspace* of a vector space V over a field $\mathbb{F}$ is any set of the form $\mathbf{a} + U := \{\mathbf{a} + \mathbf{u} \mid \mathbf{u} \in U\}$ where $\mathbf{a}$ is a vector in V and U is a linear subspace of V.

- **Remarks:**
  1. For a vector space V over a field $\mathbb{F}$:
     - every linear subspace U of V is also an affine subspace of V, since $U = \mathbf{0} + U$;
     - V is an affine subspace of itself (because V is a linear subspace of itself);
     - for every vector $\mathbf{a} \in V$, $\{\mathbf{a}\}$ is an affine subspace of V, since $\{\mathbf{a}\} = \mathbf{a} + \{\mathbf{0}\}$ and $\{\mathbf{0}\}$ is a linear subspace of V.

Definition

An *affine subspace* of a vector space V over a field $\mathbb{F}$ is any set of the form $\mathbf{a} + U := \{\mathbf{a} + \mathbf{u} \mid \mathbf{u} \in U\}$ where $\mathbf{a}$ is a vector in V and U is a linear subspace of V.

- **Remarks:**
  1. For a vector space V over a field $\mathbb{F}$:
     - every linear subspace U of V is also an affine subspace of V, since $U = \mathbf{0} + U$;
     - V is an affine subspace of itself (because V is a linear subspace of itself);
     - for every vector $\mathbf{a} \in V$, $\{\mathbf{a}\}$ is an affine subspace of V, since $\{\mathbf{a}\} = \mathbf{a} + \{\mathbf{0}\}$ and $\{\mathbf{0}\}$ is a linear subspace of V.
  2. As we know, linear subspaces of $\mathbb{R}^n$ are $\{\mathbf{0}\}$, lines through the origin, planes through the origin, and higher dimensional generalizations.

Definition

An *affine subspace* of a vector space V over a field $\mathbb{F}$ is any set of the form $\mathbf{a} + U := \{\mathbf{a} + \mathbf{u} \mid \mathbf{u} \in U\}$ where $\mathbf{a}$ is a vector in V and U is a linear subspace of V.

- **Remarks:**
  1. For a vector space V over a field $\mathbb{F}$:
     - every linear subspace U of V is also an affine subspace of V, since $U = \mathbf{0} + U$;
     - V is an affine subspace of itself (because V is a linear subspace of itself);
     - for every vector $\mathbf{a} \in V$, $\{\mathbf{a}\}$ is an affine subspace of V, since $\{\mathbf{a}\} = \mathbf{a} + \{\mathbf{0}\}$ and $\{\mathbf{0}\}$ is a linear subspace of V.
  2. As we know, linear subspaces of $\mathbb{R}^n$ are $\{\mathbf{0}\}$, lines through the origin, planes through the origin, and higher dimensional generalizations. So, affine subspaces of $\mathbb{R}^n$ are $\{\mathbf{a}\}$ (for any vector $\mathbf{a} \in \mathbb{R}^n$), lines, planes, and higher dimensional generalizations (these lines, planes, and higher dimensional generalizations may, but need not, pass through the origin).

Definition

An *affine subspace* of a vector space V over a field $\mathbb{F}$ is any set of the form $\mathbf{a} + U := \{\mathbf{a} + \mathbf{u} \mid \mathbf{u} \in U\}$ where $\mathbf{a}$ is a vector in V and U is a linear subspace of V.

- As Theorem 1.1 (next slide) shows, for an affine subspace $M = \mathbf{a} + U$ (where $\mathbf{a}$ and U are as in the definition above):
    - the vector $\mathbf{a}$ need not be unique (it can be any vector in M);
    - the linear subspace U is unique (it depends only on M, and not on the vector $\mathbf{a}$).

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(a) $\mathbf{a} \in M$ (and in particular, $M \neq \emptyset$);

(b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

(c) for all $\mathbf{b} \in V \setminus M$, we have that $M \cap (\mathbf{b} + U) = \emptyset$;

(d) for all vectors $\mathbf{a}'$ and linear subspaces U' of V s.t. $M = \mathbf{a}' + U'$, we have that $U' = U$.

- First some corollaries, then a proof.

Corollary 1.2

Let V be a vector space over a field $\mathbb{F}$. Then linear subspaces of V are precisely those affine spaces of V that contain $\mathbf{0}$. In other words, for all $U \subseteq V$, the following are equivalent:

(i) U is a linear subspace of V;

(ii) U is an affine subspace of V and $\mathbf{0} \in U$.

Proof.

Corollary 1.2

Let V be a vector space over a field $\mathbb{F}$. Then linear subspaces of V are precisely those affine spaces of V that contain $\mathbf{0}$. In other words, for all $U \subseteq V$, the following are equivalent:

(i) U is a linear subspace of V;

(ii) U is an affine subspace of V and $\mathbf{0} \in U$.

Proof. Fix $U \subseteq V$.

Corollary 1.2

Let V be a vector space over a field $\mathbb{F}$. Then linear subspaces of V are precisely those affine spaces of V that contain $\mathbf{0}$. In other words, for all $U \subseteq V$, the following are equivalent:

(i) U is a linear subspace of V;

(ii) U is an affine subspace of V and $\mathbf{0} \in U$.

Proof. Fix $U \subseteq V$. Suppose first that (i) holds.

Corollary 1.2

Let V be a vector space over a field $\mathbb{F}$. Then linear subspaces of V are precisely those affine spaces of V that contain $\mathbf{0}$. In other words, for all $U \subseteq V$, the following are equivalent:

1. (i) U is a linear subspace of V;
2. (ii) U is an affine subspace of V and $\mathbf{0} \in U$.

Proof. Fix $U \subseteq V$. Suppose first that (i) holds. Then clearly, $\mathbf{0} \in U$, and moreover, $U = \mathbf{0} + U$. So, (ii) holds.

Corollary 1.2

Let V be a vector space over a field $\mathbb{F}$. Then linear subspaces of V are precisely those affine spaces of V that contain $\mathbf{0}$. In other words, for all $U \subseteq V$, the following are equivalent:

(i) U is a linear subspace of V;

(ii) U is an affine subspace of V and $\mathbf{0} \in U$.

Proof. Fix $U \subseteq V$. Suppose first that (i) holds. Then clearly, $\mathbf{0} \in U$, and moreover, $U = \mathbf{0} + U$. So, (ii) holds.

Suppose now that (ii) holds.

Corollary 1.2

Let V be a vector space over a field $\mathbb{F}$. Then linear subspaces of V are precisely those affine spaces of V that contain $\mathbf{0}$. In other words, for all $U \subseteq V$, the following are equivalent:

(i) U is a linear subspace of V;

(ii) U is an affine subspace of V and $\mathbf{0} \in U$.

Proof. Fix $U \subseteq V$. Suppose first that (i) holds. Then clearly, $\mathbf{0} \in U$, and moreover, $U = \mathbf{0} + U$. So, (ii) holds.

Suppose now that (ii) holds. Since U is an affine subspace of V, we know that there exists a vector $\mathbf{a} \in V$ and a linear subspace U' of V s.t. $U = \mathbf{a} + U'$.

Corollary 1.2

Let V be a vector space over a field $\mathbb{F}$. Then linear subspaces of V are precisely those affine spaces of V that contain $\mathbf{0}$. In other words, for all $U \subseteq V$, the following are equivalent:

(i) U is a linear subspace of V;

(ii) U is an affine subspace of V and $\mathbf{0} \in U$.

Proof. Fix $U \subseteq V$. Suppose first that (i) holds. Then clearly, $\mathbf{0} \in U$, and moreover, $U = \mathbf{0} + U$. So, (ii) holds.

Suppose now that (ii) holds. Since U is an affine subspace of V, we know that there exists a vector $\mathbf{a} \in V$ and a linear subspace U' of V s.t. $U = \mathbf{a} + U'$. Moreover, by (ii), we have that $\mathbf{0} \in U$, and so by Theorem 1.1(b), we have that $U = \mathbf{0} + U'$.

Corollary 1.2

Let V be a vector space over a field $\mathbb{F}$. Then linear subspaces of V are precisely those affine spaces of V that contain $\mathbf{0}$. In other words, for all $U \subseteq V$, the following are equivalent:

(i) U is a linear subspace of V;

(ii) U is an affine subspace of V and $\mathbf{0} \in U$.

Proof. Fix $U \subseteq V$. Suppose first that (i) holds. Then clearly, $\mathbf{0} \in U$, and moreover, $U = \mathbf{0} + U$. So, (ii) holds.

Suppose now that (ii) holds. Since U is an affine subspace of V, we know that there exists a vector $\mathbf{a} \in V$ and a linear subspace U' of V s.t. $U = \mathbf{a} + U'$. Moreover, by (ii), we have that $\mathbf{0} \in U$, and so by Theorem 1.1(b), we have that $U = \mathbf{0} + U'$. So, $U = U'$, and so since U' is a linear subspace of V, we see that (i) holds. Q.E.D.

- Recall that the intersection of two linear subspaces is a linear subspace.

- Recall that the intersection of two linear subspaces is a linear subspace.
- In the case of affine subspaces, we have the following corollary.

- Recall that the intersection of two linear subspaces is a linear subspace.
- In the case of affine subspaces, we have the following corollary.

Corollary 1.3

Let V be a vector space over a field $\mathbb{F}$, and let M_1 and M_2 be affine subspaces of V. Then either $M_1 \cap M_2 = \emptyset$, or $M_1 \cap M_2$ is an affine subspace of V.

Proof.

- Recall that the intersection of two linear subspaces is a linear subspace.
- In the case of affine subspaces, we have the following corollary.

Corollary 1.3

Let V be a vector space over a field $\mathbb{F}$, and let M_1 and M_2 be affine subspaces of V. Then either $M_1 \cap M_2 = \emptyset$, or $M_1 \cap M_2$ is an affine subspace of V.

Proof. We may assume that $M_1 \cap M_2 \neq \emptyset$, for otherwise we are done. Fix any $\mathbf{a} \in M_1 \cap M_2$.

- Recall that the intersection of two linear subspaces is a linear subspace.
- In the case of affine subspaces, we have the following corollary.

Corollary 1.3

Let V be a vector space over a field $\mathbb{F}$, and let M_1 and M_2 be affine subspaces of V. Then either $M_1 \cap M_2 = \emptyset$, or $M_1 \cap M_2$ is an affine subspace of V.

Proof. We may assume that $M_1 \cap M_2 \neq \emptyset$, for otherwise we are done. Fix any $\mathbf{a} \in M_1 \cap M_2$. By Theorem 1.1, M_1 and M_2 can be written as $M_1 = \mathbf{a} + U_1$ and $M_2 = \mathbf{a} + U_2$, for some linear subspaces U_1 and U_2 of V.

- Recall that the intersection of two linear subspaces is a linear subspace.
- In the case of affine subspaces, we have the following corollary.

Corollary 1.3

Let V be a vector space over a field $\mathbb{F}$, and let M_1 and M_2 be affine subspaces of V. Then either $M_1 \cap M_2 = \emptyset$, or $M_1 \cap M_2$ is an affine subspace of V.

Proof. We may assume that $M_1 \cap M_2 \neq \emptyset$, for otherwise we are done. Fix any $\mathbf{a} \in M_1 \cap M_2$. By Theorem 1.1, M_1 and M_2 can be written as $M_1 = \mathbf{a} + U_1$ and $M_2 = \mathbf{a} + U_2$, for some linear subspaces U_1 and U_2 of V. Then $U := U_1 \cap U_2$ is a linear subspace of V.

- Recall that the intersection of two linear subspaces is a linear subspace.
- In the case of affine subspaces, we have the following corollary.

Corollary 1.3

Let V be a vector space over a field $\mathbb{F}$, and let M_1 and M_2 be affine subspaces of V. Then either $M_1 \cap M_2 = \emptyset$, or $M_1 \cap M_2$ is an affine subspace of V.

Proof. We may assume that $M_1 \cap M_2 \neq \emptyset$, for otherwise we are done. Fix any $\mathbf{a} \in M_1 \cap M_2$. By Theorem 1.1, M_1 and M_2 can be written as $M_1 = \mathbf{a} + U_1$ and $M_2 = \mathbf{a} + U_2$, for some linear subspaces U_1 and U_2 of V. Then $U := U_1 \cap U_2$ is a linear subspace of V. Moreover, it is clear that $M_1 \cap M_2 = \mathbf{a} + U$, and so $M_1 \cap M_2$ is an affine subspace. Q.E.D.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(a) $\mathbf{a} \in M$ (and in particular, $M \neq \emptyset$);

(b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

(c) for all $\mathbf{b} \in V \setminus M$, we have that $M \cap (\mathbf{b} + U) = \emptyset$;

(d) for all vectors $\mathbf{a}'$ and linear subspaces U' of V s.t. $M = \mathbf{a}' + U'$, we have that $U' = U$.

Proof.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(a) $\mathbf{a} \in M$ (and in particular, $M \neq \emptyset$);

(b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

(c) for all $\mathbf{b} \in V \setminus M$, we have that $M \cap (\mathbf{b} + U) = \emptyset$;

(d) for all vectors $\mathbf{a}'$ and linear subspaces U' of V s.t. $M = \mathbf{a}' + U'$, we have that $U' = U$.

Proof. (a) Since U is a linear subspace of V, we have that $\mathbf{0} \in U$, and consequently, $\mathbf{a} = \mathbf{a} + \mathbf{0} \in \mathbf{a} + U = M$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

Proof (continued). (b) Fix $\mathbf{a}' \in M$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

Proof (continued). (b) Fix $\mathbf{a}' \in M$. Since $\mathbf{a}' \in M = \mathbf{a} + U$, there exists some $\mathbf{u}' \in U$ s.t. $\mathbf{a}' = \mathbf{a} + \mathbf{u}'$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

Proof (continued). (b) Fix $\mathbf{a}' \in M$. Since $\mathbf{a}' \in M = \mathbf{a} + U$, there exists some $\mathbf{u}' \in U$ s.t. $\mathbf{a}' = \mathbf{a} + \mathbf{u}'$. WTS $M = \mathbf{a}' + U$.

Let us first show that $M \subseteq \mathbf{a}' + U$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

Proof (continued). (b) Fix $\mathbf{a}' \in M$. Since $\mathbf{a}' \in M = \mathbf{a} + U$, there exists some $\mathbf{u}' \in U$ s.t. $\mathbf{a}' = \mathbf{a} + \mathbf{u}'$. WTS $M = \mathbf{a}' + U$.

Let us first show that $M \subseteq \mathbf{a}' + U$. Fix $\mathbf{x} \in M$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

Proof (continued). (b) Fix $\mathbf{a}' \in M$. Since $\mathbf{a}' \in M = \mathbf{a} + U$, there exists some $\mathbf{u}' \in U$ s.t. $\mathbf{a}' = \mathbf{a} + \mathbf{u}'$. WTS $M = \mathbf{a}' + U$.

Let us first show that $M \subseteq \mathbf{a}' + U$. Fix $\mathbf{x} \in M$. Since $M = \mathbf{a} + U$, there exists some $\mathbf{u} \in U$ s.t. $\mathbf{x} = \mathbf{a} + \mathbf{u}$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

Proof (continued). (b) Fix $\mathbf{a}' \in M$. Since $\mathbf{a}' \in M = \mathbf{a} + U$, there exists some $\mathbf{u}' \in U$ s.t. $\mathbf{a}' = \mathbf{a} + \mathbf{u}'$. WTS $M = \mathbf{a}' + U$.

Let us first show that $M \subseteq \mathbf{a}' + U$. Fix $\mathbf{x} \in M$. Since $M = \mathbf{a} + U$, there exists some $\mathbf{u} \in U$ s.t. $\mathbf{x} = \mathbf{a} + \mathbf{u}$. Then
$\mathbf{x} = \mathbf{a} + \mathbf{u} = (\mathbf{a}' - \mathbf{u}') + \mathbf{u} = \mathbf{a}' + (\mathbf{u} - \mathbf{u}')$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

Proof (continued). (b) Fix $\mathbf{a}' \in M$. Since $\mathbf{a}' \in M = \mathbf{a} + U$, there exists some $\mathbf{u}' \in U$ s.t. $\mathbf{a}' = \mathbf{a} + \mathbf{u}'$. WTS $M = \mathbf{a}' + U$.

Let us first show that $M \subseteq \mathbf{a}' + U$. Fix $\mathbf{x} \in M$. Since $M = \mathbf{a} + U$, there exists some $\mathbf{u} \in U$ s.t. $\mathbf{x} = \mathbf{a} + \mathbf{u}$. Then $\mathbf{x} = \mathbf{a} + \mathbf{u} = (\mathbf{a}' - \mathbf{u}') + \mathbf{u} = \mathbf{a}' + (\mathbf{u} - \mathbf{u}')$. Since $\mathbf{u}, \mathbf{u}' \in U$, and U is a linear subspace of V, we have that $\mathbf{u} - \mathbf{u}' \in U$; so, $\mathbf{x} = \mathbf{a}' + (\mathbf{u} - \mathbf{u}') \in \mathbf{a}' + U$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

- (b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

Proof (continued). (b) Fix $\mathbf{a}' \in M$. Since $\mathbf{a}' \in M = \mathbf{a} + U$, there exists some $\mathbf{u}' \in U$ s.t. $\mathbf{a}' = \mathbf{a} + \mathbf{u}'$. WTS $M = \mathbf{a}' + U$.

Let us first show that $M \subseteq \mathbf{a}' + U$. Fix $\mathbf{x} \in M$. Since $M = \mathbf{a} + U$, there exists some $\mathbf{u} \in U$ s.t. $\mathbf{x} = \mathbf{a} + \mathbf{u}$. Then $\mathbf{x} = \mathbf{a} + \mathbf{u} = (\mathbf{a}' - \mathbf{u}') + \mathbf{u} = \mathbf{a}' + (\mathbf{u} - \mathbf{u}')$. Since $\mathbf{u}, \mathbf{u}' \in U$, and U is a linear subspace of V, we have that $\mathbf{u} - \mathbf{u}' \in U$; so, $\mathbf{x} = \mathbf{a}' + (\mathbf{u} - \mathbf{u}') \in \mathbf{a}' + U$. This proves that $M \subseteq \mathbf{a}' + U$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

Proof (continued). (b) Fix $\mathbf{a}' \in M$. Since $\mathbf{a}' \in M = \mathbf{a} + U$, there exists some $\mathbf{u}' \in U$ s.t. $\mathbf{a}' = \mathbf{a} + \mathbf{u}'$. WTS $M = \mathbf{a}' + U$.

Let us first show that $M \subseteq \mathbf{a}' + U$. Fix $\mathbf{x} \in M$. Since $M = \mathbf{a} + U$, there exists some $\mathbf{u} \in U$ s.t. $\mathbf{x} = \mathbf{a} + \mathbf{u}$. Then $\mathbf{x} = \mathbf{a} + \mathbf{u} = (\mathbf{a}' - \mathbf{u}') + \mathbf{u} = \mathbf{a}' + (\mathbf{u} - \mathbf{u}')$. Since $\mathbf{u}, \mathbf{u}' \in U$, and U is a linear subspace of V, we have that $\mathbf{u} - \mathbf{u}' \in U$; so, $\mathbf{x} = \mathbf{a}' + (\mathbf{u} - \mathbf{u}') \in \mathbf{a}' + U$. This proves that $M \subseteq \mathbf{a}' + U$.

Let us now show that $\mathbf{a}' + U \subseteq M$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

Proof (continued). (b) Fix $\mathbf{a}' \in M$. Since $\mathbf{a}' \in M = \mathbf{a} + U$, there exists some $\mathbf{u}' \in U$ s.t. $\mathbf{a}' = \mathbf{a} + \mathbf{u}'$. WTS $M = \mathbf{a}' + U$.

Let us first show that $M \subseteq \mathbf{a}' + U$. Fix $\mathbf{x} \in M$. Since $M = \mathbf{a} + U$, there exists some $\mathbf{u} \in U$ s.t. $\mathbf{x} = \mathbf{a} + \mathbf{u}$. Then $\mathbf{x} = \mathbf{a} + \mathbf{u} = (\mathbf{a}' - \mathbf{u}') + \mathbf{u} = \mathbf{a}' + (\mathbf{u} - \mathbf{u}')$. Since $\mathbf{u}, \mathbf{u}' \in U$, and U is a linear subspace of V, we have that $\mathbf{u} - \mathbf{u}' \in U$; so, $\mathbf{x} = \mathbf{a}' + (\mathbf{u} - \mathbf{u}') \in \mathbf{a}' + U$. This proves that $M \subseteq \mathbf{a}' + U$.

Let us now show that $\mathbf{a}' + U \subseteq M$. Fix $\mathbf{u} \in U$. WTS $\mathbf{a}' + \mathbf{u} \in M$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

Proof (continued). (b) Fix $\mathbf{a}' \in M$. Since $\mathbf{a}' \in M = \mathbf{a} + U$, there exists some $\mathbf{u}' \in U$ s.t. $\mathbf{a}' = \mathbf{a} + \mathbf{u}'$. WTS $M = \mathbf{a}' + U$.

Let us first show that $M \subseteq \mathbf{a}' + U$. Fix $\mathbf{x} \in M$. Since $M = \mathbf{a} + U$, there exists some $\mathbf{u} \in U$ s.t. $\mathbf{x} = \mathbf{a} + \mathbf{u}$. Then $\mathbf{x} = \mathbf{a} + \mathbf{u} = (\mathbf{a}' - \mathbf{u}') + \mathbf{u} = \mathbf{a}' + (\mathbf{u} - \mathbf{u}')$. Since $\mathbf{u}, \mathbf{u}' \in U$, and U is a linear subspace of V, we have that $\mathbf{u} - \mathbf{u}' \in U$; so, $\mathbf{x} = \mathbf{a}' + (\mathbf{u} - \mathbf{u}') \in \mathbf{a}' + U$. This proves that $M \subseteq \mathbf{a}' + U$.

Let us now show that $\mathbf{a}' + U \subseteq M$. Fix $\mathbf{u} \in U$. WTS $\mathbf{a}' + \mathbf{u} \in M$. But note that $\mathbf{a}' + \mathbf{u} = \mathbf{a} + \mathbf{u}' + \mathbf{u}$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

Proof (continued). (b) Fix $\mathbf{a}' \in M$. Since $\mathbf{a}' \in M = \mathbf{a} + U$, there exists some $\mathbf{u}' \in U$ s.t. $\mathbf{a}' = \mathbf{a} + \mathbf{u}'$. WTS $M = \mathbf{a}' + U$.

Let us first show that $M \subseteq \mathbf{a}' + U$. Fix $\mathbf{x} \in M$. Since $M = \mathbf{a} + U$, there exists some $\mathbf{u} \in U$ s.t. $\mathbf{x} = \mathbf{a} + \mathbf{u}$. Then
$\mathbf{x} = \mathbf{a} + \mathbf{u} = (\mathbf{a}' - \mathbf{u}') + \mathbf{u} = \mathbf{a}' + (\mathbf{u} - \mathbf{u}')$. Since $\mathbf{u}, \mathbf{u}' \in U$, and U is a linear subspace of V, we have that $\mathbf{u} - \mathbf{u}' \in U$; so,
$\mathbf{x} = \mathbf{a}' + (\mathbf{u} - \mathbf{u}') \in \mathbf{a}' + U$. This proves that $M \subseteq \mathbf{a}' + U$.

Let us now show that $\mathbf{a}' + U \subseteq M$. Fix $\mathbf{u} \in U$. WTS $\mathbf{a}' + \mathbf{u} \in M$. But note that $\mathbf{a}' + \mathbf{u} = \mathbf{a} + \mathbf{u}' + \mathbf{u}$. Since $\mathbf{u}', \mathbf{u} \in U$, and U is a linear subspace of V, we have that $\mathbf{u}' + \mathbf{u} \in U$; consequently,
$\mathbf{a}' + \mathbf{u} = \mathbf{a} + \mathbf{u}' + \mathbf{u} \in \mathbf{a} + U = M$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

Proof (continued). (b) Fix $\mathbf{a}' \in M$. Since $\mathbf{a}' \in M = \mathbf{a} + U$, there exists some $\mathbf{u}' \in U$ s.t. $\mathbf{a}' = \mathbf{a} + \mathbf{u}'$. WTS $M = \mathbf{a}' + U$.

Let us first show that $M \subseteq \mathbf{a}' + U$. Fix $\mathbf{x} \in M$. Since $M = \mathbf{a} + U$, there exists some $\mathbf{u} \in U$ s.t. $\mathbf{x} = \mathbf{a} + \mathbf{u}$. Then $\mathbf{x} = \mathbf{a} + \mathbf{u} = (\mathbf{a}' - \mathbf{u}') + \mathbf{u} = \mathbf{a}' + (\mathbf{u} - \mathbf{u}')$. Since $\mathbf{u}, \mathbf{u}' \in U$, and U is a linear subspace of V, we have that $\mathbf{u} - \mathbf{u}' \in U$; so, $\mathbf{x} = \mathbf{a}' + (\mathbf{u} - \mathbf{u}') \in \mathbf{a}' + U$. This proves that $M \subseteq \mathbf{a}' + U$.

Let us now show that $\mathbf{a}' + U \subseteq M$. Fix $\mathbf{u} \in U$. WTS $\mathbf{a}' + \mathbf{u} \in M$. But note that $\mathbf{a}' + \mathbf{u} = \mathbf{a} + \mathbf{u}' + \mathbf{u}$. Since $\mathbf{u}', \mathbf{u} \in U$, and U is a linear subspace of V, we have that $\mathbf{u}' + \mathbf{u} \in U$; consequently, $\mathbf{a}' + \mathbf{u} = \mathbf{a} + \mathbf{u}' + \mathbf{u} \in \mathbf{a} + U = M$. This proves that $\mathbf{a}' + U \subseteq M$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(c) for all $\mathbf{b} \in V \setminus M$, we have that $M \cap (\mathbf{b} + U) = \emptyset$;

Proof (continued). (c) Fix $\mathbf{b} \in V \setminus M$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(c) for all $\mathbf{b} \in V \setminus M$, we have that $M \cap (\mathbf{b} + U) = \emptyset$;

Proof (continued). (c) Fix $\mathbf{b} \in V \setminus M$. WTS $M \cap (\mathbf{b} + U) = \emptyset$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(c) for all $\mathbf{b} \in V \setminus M$, we have that $M \cap (\mathbf{b} + U) = \emptyset$;

Proof (continued). (c) Fix $\mathbf{b} \in V \setminus M$. WTS $M \cap (\mathbf{b} + U) = \emptyset$. Suppose otherwise, and fix $\mathbf{x} \in M \cap (\mathbf{b} + U)$. Since $\mathbf{x} \in M = \mathbf{a} + U$, there exists some $\mathbf{u}_1 \in U$ s.t. $\mathbf{x} = \mathbf{a} + \mathbf{u}_1$;

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(c) for all $\mathbf{b} \in V \setminus M$, we have that $M \cap (\mathbf{b} + U) = \emptyset$;

Proof (continued). (c) Fix $\mathbf{b} \in V \setminus M$. WTS $M \cap (\mathbf{b} + U) = \emptyset$. Suppose otherwise, and fix $\mathbf{x} \in M \cap (\mathbf{b} + U)$. Since $\mathbf{x} \in M = \mathbf{a} + U$, there exists some $\mathbf{u}_1 \in U$ s.t. $\mathbf{x} = \mathbf{a} + \mathbf{u}_1$; on the other hand, since $\mathbf{x} \in \mathbf{b} + U$, there exists some $\mathbf{u}_2 \in U$ s.t. $\mathbf{x} = \mathbf{b} + \mathbf{u}_2$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(c) for all $\mathbf{b} \in V \setminus M$, we have that $M \cap (\mathbf{b} + U) = \emptyset$;

Proof (continued). (c) Fix $\mathbf{b} \in V \setminus M$. WTS $M \cap (\mathbf{b} + U) = \emptyset$. Suppose otherwise, and fix $\mathbf{x} \in M \cap (\mathbf{b} + U)$. Since $\mathbf{x} \in M = \mathbf{a} + U$, there exists some $\mathbf{u}_1 \in U$ s.t. $\mathbf{x} = \mathbf{a} + \mathbf{u}_1$; on the other hand, since $\mathbf{x} \in \mathbf{b} + U$, there exists some $\mathbf{u}_2 \in U$ s.t. $\mathbf{x} = \mathbf{b} + \mathbf{u}_2$. So, $\mathbf{a} + \mathbf{u}_1 = \mathbf{b} + \mathbf{u}_2$, and it follows that $\mathbf{b} = \mathbf{a} + (\mathbf{u}_1 - \mathbf{u}_2)$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(c) for all $\mathbf{b} \in V \setminus M$, we have that $M \cap (\mathbf{b} + U) = \emptyset$;

Proof (continued). (c) Fix $\mathbf{b} \in V \setminus M$. WTS $M \cap (\mathbf{b} + U) = \emptyset$. Suppose otherwise, and fix $\mathbf{x} \in M \cap (\mathbf{b} + U)$. Since $\mathbf{x} \in M = \mathbf{a} + U$, there exists some $\mathbf{u}_1 \in U$ s.t. $\mathbf{x} = \mathbf{a} + \mathbf{u}_1$; on the other hand, since $\mathbf{x} \in \mathbf{b} + U$, there exists some $\mathbf{u}_2 \in U$ s.t. $\mathbf{x} = \mathbf{b} + \mathbf{u}_2$. So, $\mathbf{a} + \mathbf{u}_1 = \mathbf{b} + \mathbf{u}_2$, and it follows that $\mathbf{b} = \mathbf{a} + (\mathbf{u}_1 - \mathbf{u}_2)$. Since $\mathbf{u}_1, \mathbf{u}_2 \in U$, and since U is a linear subspace of V, we have that $\mathbf{u}_1 - \mathbf{u}_2 \in U$;

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(c) for all $\mathbf{b} \in V \setminus M$, we have that $M \cap (\mathbf{b} + U) = \emptyset$;

Proof (continued). (c) Fix $\mathbf{b} \in V \setminus M$. WTS $M \cap (\mathbf{b} + U) = \emptyset$. Suppose otherwise, and fix $\mathbf{x} \in M \cap (\mathbf{b} + U)$. Since $\mathbf{x} \in M = \mathbf{a} + U$, there exists some $\mathbf{u}_1 \in U$ s.t. $\mathbf{x} = \mathbf{a} + \mathbf{u}_1$; on the other hand, since $\mathbf{x} \in \mathbf{b} + U$, there exists some $\mathbf{u}_2 \in U$ s.t. $\mathbf{x} = \mathbf{b} + \mathbf{u}_2$. So, $\mathbf{a} + \mathbf{u}_1 = \mathbf{b} + \mathbf{u}_2$, and it follows that $\mathbf{b} = \mathbf{a} + (\mathbf{u}_1 - \mathbf{u}_2)$. Since $\mathbf{u}_1, \mathbf{u}_2 \in U$, and since U is a linear subspace of V, we have that $\mathbf{u}_1 - \mathbf{u}_2 \in U$; consequently, $\mathbf{b} = \mathbf{a} + (\mathbf{u}_1 - \mathbf{u}_2) \in \mathbf{a} + U = M$, contrary to the fact that $\mathbf{b} \in V \setminus M$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(a) $\mathbf{a} \in M$ (and in particular, $M \neq \emptyset$);

(b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

(c) for all $\mathbf{b} \in V \setminus M$, we have that $M \cap (\mathbf{b} + U) = \emptyset$;

(d) for all vectors $\mathbf{a}'$ and linear subspaces U' of V s.t. $M = \mathbf{a}' + U'$, we have that $U' = U$.

Proof (continued). (d) Fix a vector $\mathbf{a}'$ and a linear subspace U' of V s.t. $M = \mathbf{a}' + U'$. By (a), we have that $\mathbf{a}' \in M$, and so by (b), we have that $M = \mathbf{a}' + U$. So, $\mathbf{a}' + U' = \mathbf{a}' + U$, and we deduce that $U' = U$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(a) $\mathbf{a} \in M$ (and in particular, $M \neq \emptyset$);

(b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

(c) for all $\mathbf{b} \in V \setminus M$, we have that $M \cap (\mathbf{b} + U) = \emptyset$;

(d) for all vectors $\mathbf{a}'$ and linear subspaces U' of V s.t. $M = \mathbf{a}' + U'$, we have that $U' = U$.

Theorem 1.1

Let V be a vector space over a field $\mathbb{F}$, and let $M = \mathbf{a} + U$ be an affine subspace of V, where $\mathbf{a}$ is a vector and U a linear subspace of V. Then all the following hold:

(a) $\mathbf{a} \in M$ (and in particular, $M \neq \emptyset$);

(b) for all $\mathbf{a}' \in M$, we have that $M = \mathbf{a}' + U$;

(c) for all $\mathbf{b} \in V \setminus M$, we have that $M \cap (\mathbf{b} + U) = \emptyset$;

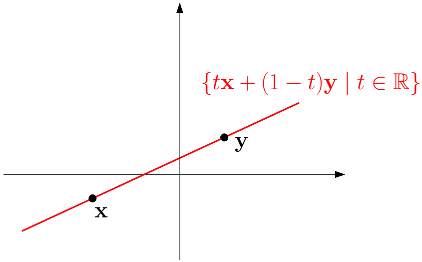
(d) for all vectors $\mathbf{a}'$ and linear subspaces U' of V s.t. $M = \mathbf{a}' + U'$, we have that $U' = U$.

- Given a vector space V over a field $\mathbb{F}$, we define the *dimension* of an affine subspace $M = \mathbf{a} + U$ of V (where $\mathbf{a}$ is a vector and U a linear subspace of V) to be $\dim(U)$.
  - By Theorem 1.1(d), this is well defined.

2. Affine combinations and affine hulls

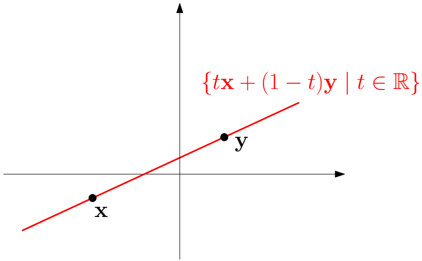
2. Affine combinations and affine hulls

- Recall from analytic geometry that if **x** and **y** are distinct points (vectors) in $\mathbb{R}^2$, then the line in $\mathbb{R}^2$ that passes through **x** and **y** is $\{t\mathbf{x} + (1 - t)\mathbf{y} \mid t \in \mathbb{R}\}$.



2. Affine combinations and affine hulls

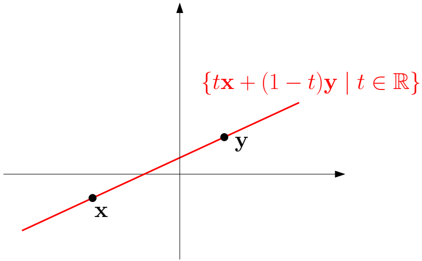
- Recall from analytic geometry that if **x** and **y** are distinct points (vectors) in $\mathbb{R}^2$, then the line in $\mathbb{R}^2$ that passes through **x** and **y** is $\{t\mathbf{x} + (1-t)\mathbf{y} \mid t \in \mathbb{R}\}$.



- This in fact holds for all distinct points **x** and **y** in $\mathbb{R}^n$ (not just $\mathbb{R}^2$).

2. Affine combinations and affine hulls

- Recall from analytic geometry that if $\mathbf{x}$ and $\mathbf{y}$ are distinct points (vectors) in $\mathbb{R}^2$, then the line in $\mathbb{R}^2$ that passes through $\mathbf{x}$ and $\mathbf{y}$ is $\{t\mathbf{x} + (1-t)\mathbf{y} \mid t \in \mathbb{R}\}$.



- This in fact holds for all distinct points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ (not just $\mathbb{R}^2$).
- Affine combinations are a generalization of this concept.

Definition

Suppose that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ ($n \geq 1$) are vectors in a vector space V over a field $\mathbb{F}$. An *affine combination* of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is any sum of the form $\alpha_1\mathbf{x}_1 + \cdots + \alpha_n\mathbf{x}_n$, where $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ satisfy $\alpha_1 + \cdots + \alpha_n = 1$. The set of all affine combinations of $\mathbf{x}_1, \ldots, \mathbf{x}_n$, denoted $\text{Aff}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, is called the *affine hull* (or *affine span*) of $\mathbf{x}_1, \ldots, \mathbf{x}_n$. So, we have that

$$\text{Aff}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \;\; := \;\; \Big\{ \sum_{i=1}^{n} \alpha_i\mathbf{x}_i \mid \alpha_1, \ldots, \alpha_n \in \mathbb{F}, \;\; \sum_{i=1}^{n} \alpha_i = 1 \Big\}.$$

Definition

Suppose that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ ($n \geq 1$) are vectors in a vector space V over a field $\mathbb{F}$. An *affine combination* of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is any sum of the form $\alpha_1 \mathbf{x}_1 + \cdots + \alpha_n \mathbf{x}_n$, where $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ satisfy $\alpha_1 + \cdots + \alpha_n = 1$. The set of all affine combinations of $\mathbf{x}_1, \ldots, \mathbf{x}_n$, denoted $\text{Aff}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, is called the *affine hull* (or *affine span*) of $\mathbf{x}_1, \ldots, \mathbf{x}_n$. So, we have that

$$\text{Aff}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \;\; := \;\; \Big\{ \sum_{i=1}^{n} \alpha_i \mathbf{x}_i \mid \alpha_1, \ldots, \alpha_n \in \mathbb{F}, \;\; \sum_{i=1}^{n} \alpha_i = 1 \Big\}.$$

- Since $\mathbf{x}_i = 0\mathbf{x}_1 + \cdots + 0\mathbf{x}_{i-1} + 1\mathbf{x}_i + 0\mathbf{x}_{i+1} + \cdots + 0\mathbf{x}_n$ for all $i \in \{1, \ldots, n\}$, we see that $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \text{Aff}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$.

Definition

Suppose that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ ($n \geq 1$) are vectors in a vector space V over a field $\mathbb{F}$. An *affine combination* of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is any sum of the form $\alpha_1\mathbf{x}_1 + \cdots + \alpha_n\mathbf{x}_n$, where $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ satisfy $\alpha_1 + \cdots + \alpha_n = 1$. The set of all affine combinations of $\mathbf{x}_1, \ldots, \mathbf{x}_n$, denoted $\text{Aff}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, is called the *affine hull* (or *affine span*) of $\mathbf{x}_1, \ldots, \mathbf{x}_n$. So, we have that

$$\text{Aff}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \quad := \quad \Big\{ \sum_{i=1}^{n} \alpha_i \mathbf{x}_i \mid \alpha_1, \ldots, \alpha_n \in \mathbb{F}, \ \sum_{i=1}^{n} \alpha_i = 1 \Big\}.$$

- Since $\mathbf{x}_i = 0\mathbf{x}_1 + \cdots + 0\mathbf{x}_{i-1} + 1\mathbf{x}_i + 0\mathbf{x}_{i+1} + \cdots + 0\mathbf{x}_n$ for all $i \in \{1, \ldots, n\}$, we see that $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \text{Aff}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$.
- As Theorem 2.1 (next slide) shows, affine subspaces of V are precisely those non-empty subsets of V that are closed under affine combinations.

Definition

Suppose that $\mathbf{x}_1, \dots, \mathbf{x}_n$ ($n \geq 1$) are vectors in a vector space V over a field $\mathbb{F}$. An *affine combination* of $\mathbf{x}_1, \dots, \mathbf{x}_n$ is any sum of the form $\alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n$, where $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ satisfy $\alpha_1 + \dots + \alpha_n = 1$. The set of all affine combinations of $\mathbf{x}_1, \dots, \mathbf{x}_n$, denoted $\text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$, is called the *affine hull* (or *affine span*) of $\mathbf{x}_1, \dots, \mathbf{x}_n$. So, we have that

$$\text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n) \quad := \quad \Big\{ \sum_{i=1}^{n} \alpha_i\mathbf{x}_i \mid \alpha_1, \dots, \alpha_n \in \mathbb{F}, \ \ \sum_{i=1}^{n} \alpha_i = 1 \Big\}.$$

- Since $\mathbf{x}_i = 0\mathbf{x}_1 + \dots + 0\mathbf{x}_{i-1} + 1\mathbf{x}_i + 0\mathbf{x}_{i+1} + \dots + 0\mathbf{x}_n$ for all $i \in \{1, \dots, n\}$, we see that $\mathbf{x}_1, \dots, \mathbf{x}_n \in \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$.
- As Theorem 2.1 (next slide) shows, affine subspaces of V are precisely those non-empty subsets of V that are closed under affine combinations.
- As a corollary (see Corollary 2.2), we deduce that all affine hulls are affine subspaces of V.

Theorem 2.1

Let V be a vector space over a field $\mathbb{F}$, and let M be a **non-empty** subset of V. Then the following are equivalent:

1. (i) M is an affine subspace of V;
2. (ii) M is closed under affine combinations, that is, for all $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$ and $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_1 + \dots + \alpha_n = 1$, we have that $\alpha_1 \mathbf{x}_1 + \dots + \alpha_n \mathbf{x}_n \in M$.

1. M is an affine subspace of V;
2. M is closed under affine combinations.

1. (i) M is an affine subspace of V;
2. (ii) M is closed under affine combinations.

Proof.

1. M is an affine subspace of V;
2. M is closed under affine combinations.

Proof. "(i) $\implies$ (ii)":

1. (i) M is an affine subspace of V;
2. (ii) M is closed under affine combinations.

Proof. "(i) $\Longrightarrow$ (ii)": Set $M = \mathbf{a} + U$, where $\mathbf{a}$ is a vector and U a linear subspace of V, as in the definition of an affine subspace.

(i) M is an affine subspace of V;

(ii) M is closed under affine combinations.

Proof. "(i) $\implies$ (ii)": Set $M = \mathbf{a} + U$, where $\mathbf{a}$ is a vector and U a linear subspace of V, as in the definition of an affine subspace. Fix $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, and fix $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_1 + \dots + \alpha_n = 1$; WTS $\alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n \in M$.

1. (i) M is an affine subspace of V;
2. (ii) M is closed under affine combinations.

Proof. "(i) $\implies$ (ii)": Set $M = \mathbf{a} + U$, where $\mathbf{a}$ is a vector and U a linear subspace of V, as in the definition of an affine subspace. Fix $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, and fix $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_1 + \dots + \alpha_n = 1$; WTS $\alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n \in M$.

Since $\mathbf{x}_1, \dots, \mathbf{x}_n \in M = \mathbf{a} + U$, there exist vectors $\mathbf{u}_1, \dots, \mathbf{u}_n \in U$ s.t. $\mathbf{x}_1 = \mathbf{a} + \mathbf{u}_1, \dots, \mathbf{x}_n = \mathbf{a} + \mathbf{u}_n$.

1. M is an affine subspace of V;
2. M is closed under affine combinations.

Proof. "(i) $\Longrightarrow$ (ii)": Set $M = \mathbf{a} + U$, where $\mathbf{a}$ is a vector and U a linear subspace of V, as in the definition of an affine subspace. Fix $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, and fix $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_1 + \dots + \alpha_n = 1$; WTS $\alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n \in M$.

Since $\mathbf{x}_1, \dots, \mathbf{x}_n \in M = \mathbf{a} + U$, there exist vectors $\mathbf{u}_1, \dots, \mathbf{u}_n \in U$ s.t. $\mathbf{x}_1 = \mathbf{a} + \mathbf{u}_1, \dots, \mathbf{x}_n = \mathbf{a} + \mathbf{u}_n$. We now have that

$$\begin{aligned}
\alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n &= \alpha_1(\mathbf{a} + \mathbf{u}_1) + \dots + \alpha_n(\mathbf{a} + \mathbf{u}_1) \\
&= (\underbrace{\alpha_1 + \dots + \alpha_n}_{=1})\mathbf{a} + (\alpha_1\mathbf{u}_1 + \dots + \alpha_n\mathbf{u}_n) \\
&= \mathbf{a} + (\underbrace{\alpha_1\mathbf{u}_1 + \dots + \alpha_n\mathbf{u}_n}_{:=\mathbf{u}}).
\end{aligned}$$

1. M is an affine subspace of V;
2. M is closed under affine combinations.

Proof. "(i) $\Longrightarrow$ (ii)": Set $M = \mathbf{a} + U$, where $\mathbf{a}$ is a vector and U a linear subspace of V, as in the definition of an affine subspace. Fix $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, and fix $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_1 + \dots + \alpha_n = 1$; WTS $\alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n \in M$.

Since $\mathbf{x}_1, \dots, \mathbf{x}_n \in M = \mathbf{a} + U$, there exist vectors $\mathbf{u}_1, \dots, \mathbf{u}_n \in U$ s.t. $\mathbf{x}_1 = \mathbf{a} + \mathbf{u}_1, \dots, \mathbf{x}_n = \mathbf{a} + \mathbf{u}_n$. We now have that

$$\begin{aligned} \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n &= \alpha_1(\mathbf{a} + \mathbf{u}_1) + \dots + \alpha_n(\mathbf{a} + \mathbf{u}_1) \\ &= \underbrace{(\alpha_1 + \dots + \alpha_n)}_{=1}\mathbf{a} + (\alpha_1\mathbf{u}_1 + \dots + \alpha_n\mathbf{u}_n) \\ &= \mathbf{a} + \underbrace{(\alpha_1\mathbf{u}_1 + \dots + \alpha_n\mathbf{u}_n)}_{:=\mathbf{u}}. \end{aligned}$$

Since $\mathbf{u}_1, \dots, \mathbf{u}_n \in U$, and U is a linear subspace of V, we have that $\mathbf{u} \in U$.

1. M is an affine subspace of V;
2. M is closed under affine combinations.

Proof. "(i) $\implies$ (ii)": Set $M = \mathbf{a} + U$, where $\mathbf{a}$ is a vector and U a linear subspace of V, as in the definition of an affine subspace. Fix $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, and fix $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_1 + \dots + \alpha_n = 1$; WTS $\alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n \in M$.

Since $\mathbf{x}_1, \dots, \mathbf{x}_n \in M = \mathbf{a} + U$, there exist vectors $\mathbf{u}_1, \dots, \mathbf{u}_n \in U$ s.t. $\mathbf{x}_1 = \mathbf{a} + \mathbf{u}_1, \dots, \mathbf{x}_n = \mathbf{a} + \mathbf{u}_n$. We now have that

$$\begin{aligned} \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n &= \alpha_1(\mathbf{a} + \mathbf{u}_1) + \dots + \alpha_n(\mathbf{a} + \mathbf{u}_1) \\ &= (\underbrace{\alpha_1 + \dots + \alpha_n}_{=1})\mathbf{a} + (\alpha_1\mathbf{u}_1 + \dots + \alpha_n\mathbf{u}_n) \\ &= \mathbf{a} + (\underbrace{\alpha_1\mathbf{u}_1 + \dots + \alpha_n\mathbf{u}_n}_{:=\mathbf{u}}). \end{aligned}$$

Since $\mathbf{u}_1, \dots, \mathbf{u}_n \in U$, and U is a linear subspace of V, we have that $\mathbf{u} \in U$. So, $\alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n = \mathbf{a} + \mathbf{u} \in \mathbf{a} + U = M$, i.e. (ii) holds.

1. M is an affine subspace of V;
2. M is closed under affine combinations.

Proof (continued). "(ii) $\implies$ (i)":

1. M is an affine subspace of V;
2. M is closed under affine combinations.

Proof (continued). "(ii) $\implies$ (i)": Using the fact that $M \neq \emptyset$, we fix some $\mathbf{a} \in M$.

1. (i) M is an affine subspace of V;
2. (ii) M is closed under affine combinations.

Proof (continued). "(ii) $\implies$ (i)": Using the fact that $M \neq \emptyset$, we fix some $\mathbf{a} \in M$. Set $U := \{\mathbf{x} - \mathbf{a} \mid \mathbf{x} \in M\}$.

1. (i) M is an affine subspace of V;
2. (ii) M is closed under affine combinations.

Proof (continued). "(ii) $\Longrightarrow$ (i)": Using the fact that $M \neq \emptyset$, we fix some $\mathbf{a} \in M$. Set $U := \{\mathbf{x} - \mathbf{a} \mid \mathbf{x} \in M\}$. Clearly, $M = \mathbf{a} + U$.

(i) M is an affine subspace of V;

(ii) M is closed under affine combinations.

Proof (continued). "(ii) $\implies$ (i)": Using the fact that $M \neq \emptyset$, we fix some $\mathbf{a} \in M$. Set $U := \{\mathbf{x} - \mathbf{a} \mid \mathbf{x} \in M\}$. Clearly, $M = \mathbf{a} + U$. It remains to show that U is a linear subspace of V.

1. M is an affine subspace of V;
2. M is closed under affine combinations.

Proof (continued). "(ii) $\Longrightarrow$ (i)": Using the fact that $M \neq \emptyset$, we fix some $\mathbf{a} \in M$. Set $U := \{\mathbf{x} - \mathbf{a} \mid \mathbf{x} \in M\}$. Clearly, $M = \mathbf{a} + U$. It remains to show that U is a linear subspace of V. By Theorem 2.7 of Lecture Notes 6, it suffices to show that $\mathbf{0} \in U$, and that U is closed under vector addition and scalar multiplication.

(i) M is an affine subspace of V;

(ii) M is closed under affine combinations.

Proof (continued). "(ii) $\Longrightarrow$ (i)": Using the fact that $M \neq \emptyset$, we fix some $\mathbf{a} \in M$. Set $U := \{\mathbf{x} - \mathbf{a} \mid \mathbf{x} \in M\}$. Clearly, $M = \mathbf{a} + U$. It remains to show that U is a linear subspace of V. By Theorem 2.7 of Lecture Notes 6, it suffices to show that $\mathbf{0} \in U$, and that U is closed under vector addition and scalar multiplication.

First, since $\mathbf{a} \in M$, we have that $\mathbf{0} = \mathbf{a} - \mathbf{a} \in U$.

1. M is an affine subspace of V;
2. M is closed under affine combinations.

Proof (continued). "(ii) $\Longrightarrow$ (i)": Using the fact that $M \neq \emptyset$, we fix some $\mathbf{a} \in M$. Set $U := \{\mathbf{x} - \mathbf{a} \mid \mathbf{x} \in M\}$. Clearly, $M = \mathbf{a} + U$. It remains to show that U is a linear subspace of V. By Theorem 2.7 of Lecture Notes 6, it suffices to show that $\mathbf{0} \in U$, and that U is closed under vector addition and scalar multiplication.

First, since $\mathbf{a} \in M$, we have that $\mathbf{0} = \mathbf{a} - \mathbf{a} \in U$.

Next, fix $\mathbf{u}_1, \mathbf{u}_2 \in U$. WTS $\mathbf{u}_1 + \mathbf{u}_2 \in U$.

1. M is an affine subspace of V;
2. M is closed under affine combinations.

Proof (continued). "(ii) $\Longrightarrow$ (i)": Using the fact that $M \neq \emptyset$, we fix some $\mathbf{a} \in M$. Set $U := \{\mathbf{x} - \mathbf{a} \mid \mathbf{x} \in M\}$. Clearly, $M = \mathbf{a} + U$. It remains to show that U is a linear subspace of V. By Theorem 2.7 of Lecture Notes 6, it suffices to show that $\mathbf{0} \in U$, and that U is closed under vector addition and scalar multiplication.

First, since $\mathbf{a} \in M$, we have that $\mathbf{0} = \mathbf{a} - \mathbf{a} \in U$.

Next, fix $\mathbf{u}_1, \mathbf{u}_2 \in U$. WTS $\mathbf{u}_1 + \mathbf{u}_2 \in U$. Since $\mathbf{u}_1, \mathbf{u}_2 \in U$, there exist $\mathbf{x}_1, \mathbf{x}_2 \in M$ s.t. $\mathbf{u}_1 = \mathbf{x}_1 - \mathbf{a}$ and $\mathbf{u}_2 = \mathbf{x}_2 - \mathbf{a}$.

1. M is an affine subspace of V;
2. M is closed under affine combinations.

Proof (continued). "(ii) $\Longrightarrow$ (i)": Using the fact that $M \neq \emptyset$, we fix some $\mathbf{a} \in M$. Set $U := \{\mathbf{x} - \mathbf{a} \mid \mathbf{x} \in M\}$. Clearly, $M = \mathbf{a} + U$. It remains to show that U is a linear subspace of V. By Theorem 2.7 of Lecture Notes 6, it suffices to show that $\mathbf{0} \in U$, and that U is closed under vector addition and scalar multiplication.

First, since $\mathbf{a} \in M$, we have that $\mathbf{0} = \mathbf{a} - \mathbf{a} \in U$.

Next, fix $\mathbf{u}_1, \mathbf{u}_2 \in U$. WTS $\mathbf{u}_1 + \mathbf{u}_2 \in U$. Since $\mathbf{u}_1, \mathbf{u}_2 \in U$, there exist $\mathbf{x}_1, \mathbf{x}_2 \in M$ s.t. $\mathbf{u}_1 = \mathbf{x}_1 - \mathbf{a}$ and $\mathbf{u}_2 = \mathbf{x}_2 - \mathbf{a}$. Then

$$\mathbf{u}_1 + \mathbf{u}_2 \quad = \quad (\mathbf{x}_1 - \mathbf{a}) + (\mathbf{x}_2 - \mathbf{a}) \quad = \quad \Big(\underbrace{1\mathbf{x}_1 + 1\mathbf{x}_2 + (-1)\mathbf{a}}_{:=\mathbf{y}} \Big) - \mathbf{a}.$$

1. M is an affine subspace of V;
2. M is closed under affine combinations.

Proof (continued). "(ii) $\implies$ (i)": Using the fact that $M \neq \emptyset$, we fix some $\mathbf{a} \in M$. Set $U := \{\mathbf{x} - \mathbf{a} \mid \mathbf{x} \in M\}$. Clearly, $M = \mathbf{a} + U$. It remains to show that U is a linear subspace of V. By Theorem 2.7 of Lecture Notes 6, it suffices to show that $\mathbf{0} \in U$, and that U is closed under vector addition and scalar multiplication.

First, since $\mathbf{a} \in M$, we have that $\mathbf{0} = \mathbf{a} - \mathbf{a} \in U$.

Next, fix $\mathbf{u}_1, \mathbf{u}_2 \in U$. WTS $\mathbf{u}_1 + \mathbf{u}_2 \in U$. Since $\mathbf{u}_1, \mathbf{u}_2 \in U$, there exist $\mathbf{x}_1, \mathbf{x}_2 \in M$ s.t. $\mathbf{u}_1 = \mathbf{x}_1 - \mathbf{a}$ and $\mathbf{u}_2 = \mathbf{x}_2 - \mathbf{a}$. Then

$$\mathbf{u}_1 + \mathbf{u}_2 \quad = \quad (\mathbf{x}_1 - \mathbf{a}) + (\mathbf{x}_2 - \mathbf{a}) \quad = \quad \Big(\underbrace{1\mathbf{x}_1 + 1\mathbf{x}_2 + (-1)\mathbf{a}}_{:=\mathbf{y}} \Big) - \mathbf{a}.$$

Since $\mathbf{x}_1, \mathbf{x}_2, \mathbf{a} \in M$ and $1 + 1 + (-1) = 1$, and since (ii) holds, we see that $\mathbf{y} \in M$. But now $\mathbf{u}_1 + \mathbf{u}_2 = \mathbf{y} - \mathbf{a} \in U$.

1. (i) M is an affine subspace of V;
2. (ii) M is closed under affine combinations.

Proof (continued). "(ii) $\implies$ (i)": Reminder $U := \{\mathbf{x} - \mathbf{a} \mid \mathbf{x} \in M\}$.

1. M is an affine subspace of V;
2. M is closed under affine combinations.

Proof (continued). "(ii) $\implies$ (i)": Reminder $U := \{\mathbf{x} - \mathbf{a} \mid \mathbf{x} \in M\}$.

Finally, fix $\mathbf{u} \in U$ and $\alpha \in \mathbb{F}$; WTS $\alpha\mathbf{u} \in U$.

1. M is an affine subspace of V;
2. M is closed under affine combinations.

Proof (continued). "(ii) $\Longrightarrow$ (i)": Reminder $U := \{\mathbf{x} - \mathbf{a} \mid \mathbf{x} \in M\}$.

Finally, fix $\mathbf{u} \in U$ and $\alpha \in \mathbb{F}$; WTS $\alpha\mathbf{u} \in U$. Since $\mathbf{u} \in U$, we know that there exists some $\mathbf{x} \in M$ s.t. $\mathbf{u} = \mathbf{x} - \mathbf{a}$.

1. M is an affine subspace of V;
2. M is closed under affine combinations.

Proof (continued). "(ii) $\Longrightarrow$ (i)": Reminder $U := \{\mathbf{x} - \mathbf{a} \mid \mathbf{x} \in M\}$.

Finally, fix $\mathbf{u} \in U$ and $\alpha \in \mathbb{F}$; WTS $\alpha\mathbf{u} \in U$. Since $\mathbf{u} \in U$, we know that there exists some $\mathbf{x} \in M$ s.t. $\mathbf{u} = \mathbf{x} - \mathbf{a}$. But now

$$\alpha\mathbf{u} \;=\; \alpha(\mathbf{x} - \mathbf{a}) \;=\; \Big(\underbrace{\alpha\mathbf{x} + (1-\alpha)\mathbf{a}}_{:=\mathbf{y}}\Big) - \mathbf{a}.$$

1. M is an affine subspace of V;
2. M is closed under affine combinations.

Proof (continued). "(ii) $\implies$ (i)": Reminder $U := \{\mathbf{x} - \mathbf{a} \mid \mathbf{x} \in M\}$.

Finally, fix $\mathbf{u} \in U$ and $\alpha \in \mathbb{F}$; WTS $\alpha\mathbf{u} \in U$. Since $\mathbf{u} \in U$, we know that there exists some $\mathbf{x} \in M$ s.t. $\mathbf{u} = \mathbf{x} - \mathbf{a}$. But now

$$\alpha\mathbf{u} = \alpha(\mathbf{x} - \mathbf{a}) = \Big(\underbrace{\alpha\mathbf{x} + (1-\alpha)\mathbf{a}}_{:=\mathbf{y}}\Big) - \mathbf{a}.$$

Since $\mathbf{x}, \mathbf{a} \in M$, and since (ii) holds, we have that $\mathbf{y} = \alpha\mathbf{x} + (1-\alpha)\mathbf{a} \in M$.

1. M is an affine subspace of V;
2. M is closed under affine combinations.

Proof (continued). "(ii) $\Longrightarrow$ (i)": Reminder $U := \{\mathbf{x} - \mathbf{a} \mid \mathbf{x} \in M\}$.

Finally, fix $\mathbf{u} \in U$ and $\alpha \in \mathbb{F}$; WTS $\alpha\mathbf{u} \in U$. Since $\mathbf{u} \in U$, we know that there exists some $\mathbf{x} \in M$ s.t. $\mathbf{u} = \mathbf{x} - \mathbf{a}$. But now

$$\alpha\mathbf{u} \;=\; \alpha(\mathbf{x} - \mathbf{a}) \;=\; \Big(\underbrace{\alpha\mathbf{x} + (1-\alpha)\mathbf{a}}_{:=\mathbf{y}}\Big) - \mathbf{a}.$$

Since $\mathbf{x}, \mathbf{a} \in M$, and since (ii) holds, we have that $\mathbf{y} = \alpha\mathbf{x} + (1-\alpha)\mathbf{a} \in M$. But now $\alpha\mathbf{u} = \mathbf{y} - \mathbf{a} \in U$. Q.E.D.

Theorem 2.1

Let V be a vector space over a field $\mathbb{F}$, and let M be a **non-empty** subset of V. Then the following are equivalent:

(i) M is an affine subspace of V;

(ii) M is closed under affine combinations, that is, for all $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$ and $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_1 + \dots + \alpha_n = 1$, we have that $\alpha_1 \mathbf{x}_1 + \dots + \alpha_n \mathbf{x}_n \in M$.

Theorem 2.1

Let V be a vector space over a field $\mathbb{F}$, and let M be a **non-empty** subset of V. Then the following are equivalent:

1. (i) M is an affine subspace of V;
2. (ii) M is closed under affine combinations, that is, for all $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$ and $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_1 + \dots + \alpha_n = 1$, we have that $\alpha_1 \mathbf{x}_1 + \dots + \alpha_n \mathbf{x}_n \in M$.

Corollary 2.2

Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ $(n \geq 1)$ be vectors in a vector space V over a field $\mathbb{F}$. Then $M := \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is an affine subspace of V.

Corollary 2.2

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ $(n \geq 1)$ be vectors in a vector space V over a field $\mathbb{F}$. Then $M := \mathrm{Aff}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is an affine subspace of V.

Proof.

Corollary 2.2

Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ $(n \geq 1)$ be vectors in a vector space V over a field $\mathbb{F}$. Then $M := \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is an affine subspace of V.

Proof. Since $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, we see that $M \neq \emptyset$. In view of Theorem 2.1, it now suffices to show that M is closed under affine combinations.

Corollary 2.2

Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ ($n \geq 1$) be vectors in a vector space V over a field $\mathbb{F}$. Then $M := \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is an affine subspace of V.

Proof. Since $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, we see that $M \neq \emptyset$. In view of Theorem 2.1, it now suffices to show that M is closed under affine combinations. Fix $\mathbf{y}_1, \dots, \mathbf{y}_m \in M$ and $\alpha_1, \dots, \alpha_m \in \mathbb{F}$ s.t. $\alpha_1 + \dots + \alpha_m = 1$. WTS $\mathbf{y} := \alpha_1 \mathbf{y}_1 + \dots + \alpha_m \mathbf{y}_m$ belongs to M.

Corollary 2.2

Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ $(n \geq 1)$ be vectors in a vector space V over a field $\mathbb{F}$. Then $M := \mathrm{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is an affine subspace of V.

Proof. Since $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, we see that $M \neq \emptyset$. In view of Theorem 2.1, it now suffices to show that M is closed under affine combinations. Fix $\mathbf{y}_1, \dots, \mathbf{y}_m \in M$ and $\alpha_1, \dots, \alpha_m \in \mathbb{F}$ s.t. $\alpha_1 + \dots + \alpha_m = 1$. WTS $\mathbf{y} := \alpha_1 \mathbf{y}_1 + \dots + \alpha_m \mathbf{y}_m$ belongs to M. Since $\mathbf{y}_1, \dots, \mathbf{y}_m \in M$, we see that for all $i \in \{1, \dots, m\}$, there exist scalars $\beta_{i,1}, \dots, \beta_{i,n} \in \mathbb{F}$ s.t. $\mathbf{y}_i = \sum_{j=1}^n \beta_{i,j} \mathbf{x}_j$ and $\sum_{j=1}^n \beta_{i,j} = 1$.

Corollary 2.2

Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ ($n \geq 1$) be vectors in a vector space V over a field $\mathbb{F}$. Then $M := \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is an affine subspace of V.

Proof. Since $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, we see that $M \neq \emptyset$. In view of Theorem 2.1, it now suffices to show that M is closed under affine combinations. Fix $\mathbf{y}_1, \dots, \mathbf{y}_m \in M$ and $\alpha_1, \dots, \alpha_m \in \mathbb{F}$ s.t. $\alpha_1 + \dots + \alpha_m = 1$. WTS $\mathbf{y} := \alpha_1 \mathbf{y}_1 + \dots + \alpha_m \mathbf{y}_m$ belongs to M. Since $\mathbf{y}_1, \dots, \mathbf{y}_m \in M$, we see that for all $i \in \{1, \dots, m\}$, there exist scalars $\beta_{i,1}, \dots, \beta_{i,n} \in \mathbb{F}$ s.t. $\mathbf{y}_i = \sum_{j=1}^n \beta_{i,j} \mathbf{x}_j$ and $\sum_{j=1}^n \beta_{i,j} = 1$. But now:

- $\mathbf{y} = \sum\limits_{i=1}^m \alpha_i \mathbf{y}_i = \sum\limits_{i=1}^m \alpha_i \Big(\sum\limits_{j=1}^n \beta_{i,j} \mathbf{x}_j \Big) = \sum\limits_{j=1}^n \sum\limits_{i=1}^m \alpha_i \beta_{i,j} \mathbf{x}_j,$

Corollary 2.2

Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ $(n \geq 1)$ be vectors in a vector space V over a field $\mathbb{F}$. Then $M := \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is an affine subspace of V.

Proof. Since $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, we see that $M \neq \emptyset$. In view of Theorem 2.1, it now suffices to show that M is closed under affine combinations. Fix $\mathbf{y}_1, \dots, \mathbf{y}_m \in M$ and $\alpha_1, \dots, \alpha_m \in \mathbb{F}$ s.t. $\alpha_1 + \dots + \alpha_m = 1$. WTS $\mathbf{y} := \alpha_1 \mathbf{y}_1 + \dots + \alpha_m \mathbf{y}_m$ belongs to M. Since $\mathbf{y}_1, \dots, \mathbf{y}_m \in M$, we see that for all $i \in \{1, \dots, m\}$, there exist scalars $\beta_{i,1}, \dots, \beta_{i,n} \in \mathbb{F}$ s.t. $\mathbf{y}_i = \sum_{j=1}^n \beta_{i,j} \mathbf{x}_j$ and $\sum_{j=1}^n \beta_{i,j} = 1$. But now:

- $\mathbf{y} = \sum_{i=1}^m \alpha_i \mathbf{y}_i = \sum_{i=1}^m \alpha_i \Big(\sum_{j=1}^n \beta_{i,j} \mathbf{x}_j \Big) = \sum_{j=1}^n \sum_{i=1}^m \alpha_i \beta_{i,j} \mathbf{x}_j$,
- $\sum_{j=1}^n \sum_{i=1}^m \alpha_i \beta_{i,j} = \sum_{i=1}^m \alpha_i \Big(\sum_{j=1}^n \beta_{i,j} \Big) \stackrel{(*)}{=} \sum_{i=1}^m \alpha_i = 1$,

where where (*) follows from the fact that $\sum_{j=1}^n \beta_{i,j} = 1$.

Corollary 2.2

Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ ($n \geq 1$) be vectors in a vector space V over a field $\mathbb{F}$. Then $M := \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is an affine subspace of V.

Proof. Since $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, we see that $M \neq \emptyset$. In view of Theorem 2.1, it now suffices to show that M is closed under affine combinations. Fix $\mathbf{y}_1, \dots, \mathbf{y}_m \in M$ and $\alpha_1, \dots, \alpha_m \in \mathbb{F}$ s.t. $\alpha_1 + \dots + \alpha_m = 1$. WTS $\mathbf{y} := \alpha_1 \mathbf{y}_1 + \dots + \alpha_m \mathbf{y}_m$ belongs to M. Since $\mathbf{y}_1, \dots, \mathbf{y}_m \in M$, we see that for all $i \in \{1, \dots, m\}$, there exist scalars $\beta_{i,1}, \dots, \beta_{i,n} \in \mathbb{F}$ s.t. $\mathbf{y}_i = \sum_{j=1}^n \beta_{i,j} \mathbf{x}_j$ and $\sum_{j=1}^n \beta_{i,j} = 1$. But now:

- $\mathbf{y} = \sum\limits_{i=1}^m \alpha_i \mathbf{y}_i = \sum\limits_{i=1}^m \alpha_i \Big(\sum\limits_{j=1}^n \beta_{i,j} \mathbf{x}_j \Big) = \sum\limits_{j=1}^n \sum\limits_{i=1}^m \alpha_i \beta_{i,j} \mathbf{x}_j,$
- $\sum\limits_{j=1}^n \sum\limits_{i=1}^m \alpha_i \beta_{i,j} = \sum\limits_{i=1}^m \alpha_i \Big(\sum\limits_{j=1}^n \beta_{i,j} \Big) \overset{(*)}{=} \sum\limits_{i=1}^m \alpha_i = 1,$

where where (*) follows from the fact that $\sum_{j=1}^n \beta_{i,j} = 1$. This proves that $\mathbf{y}$ is an affine combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$, and so $\mathbf{y} \in M$. Q.E.D.

Theorem 2.1

Let V be a vector space over a field $\mathbb{F}$, and let M be a **non-empty** subset of V. Then the following are equivalent:

(i) M is an affine subspace of V;

(ii) M is closed under affine combinations, that is, for all $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$ and $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_1 + \dots + \alpha_n = 1$, we have that $\alpha_1 \mathbf{x}_1 + \dots + \alpha_n \mathbf{x}_n \in M$.

Corollary 2.3

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_1, \dots, \mathbf{x}_n$ ($n \geq 1$) be vectors in V. Then the following are equivalent:

(i) $M = \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$;

(ii) $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, and every vector in M is an affine combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$.

Corollary 2.3

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_1, \dots, \mathbf{x}_n$ $(n \geq 1)$ be vectors in V. Then the following are equivalent:

1. (i) $M = \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$;
2. (ii) $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, and every vector in M is an affine combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$.

Proof.

Corollary 2.3

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_1, \dots, \mathbf{x}_n$ $(n \geq 1)$ be vectors in V. Then the following are equivalent:

1. (i) $M = \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$;
2. (ii) $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, and every vector in M is an affine combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$.

Proof. Obviously, (i) implies (ii). For the reverse implication, we assume that (ii) holds, and we prove (i).

Corollary 2.3

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_1, \dots, \mathbf{x}_n$ ($n \geq 1$) be vectors in V. Then the following are equivalent:

1. (i) $M = \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$;
2. (ii) $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, and every vector in M is an affine combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$.

Proof. Obviously, (i) implies (ii). For the reverse implication, we assume that (ii) holds, and we prove (i). Since every vector in M is an affine combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$, we have that $M \subseteq \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$.

Corollary 2.3

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_1, \dots, \mathbf{x}_n$ $(n \geq 1)$ be vectors in V. Then the following are equivalent:

1. (i) $M = \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$;
2. (ii) $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, and every vector in M is an affine combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$.

Proof. Obviously, (i) implies (ii). For the reverse implication, we assume that (ii) holds, and we prove (i). Since every vector in M is an affine combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$, we have that $M \subseteq \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$. WTS $\text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n) \subseteq M$.

Corollary 2.3

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_1, \dots, \mathbf{x}_n$ ($n \geq 1$) be vectors in V. Then the following are equivalent:

1. (i) $M = \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$;
2. (ii) $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, and every vector in M is an affine combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$.

Proof. Obviously, (i) implies (ii). For the reverse implication, we assume that (ii) holds, and we prove (i). Since every vector in M is an affine combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$, we have that $M \subseteq \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$. WTS $\text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n) \subseteq M$. Fix $\mathbf{x} \in \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$. By (ii), we have that $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, and by Theorem 2.1, we know that M is closed under affine combinations.

Corollary 2.3

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_1, \dots, \mathbf{x}_n$ ($n \geq 1$) be vectors in V. Then the following are equivalent:

1. (i) $M = \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$;
2. (ii) $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, and every vector in M is an affine combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$.

Proof. Obviously, (i) implies (ii). For the reverse implication, we assume that (ii) holds, and we prove (i). Since every vector in M is an affine combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$, we have that $M \subseteq \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$. WTS $\text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n) \subseteq M$. Fix $\mathbf{x} \in \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$. By (ii), we have that $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, and by Theorem 2.1, we know that M is closed under affine combinations. Since $\mathbf{x}$ is an affine combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$, we deduce that $\mathbf{x} \in M$.

Corollary 2.3

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_1, \dots, \mathbf{x}_n$ $(n \geq 1)$ be vectors in V. Then the following are equivalent:

1. (i) $M = \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$;
2. (ii) $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, and every vector in M is an affine combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$.

Proof. Obviously, (i) implies (ii). For the reverse implication, we assume that (ii) holds, and we prove (i). Since every vector in M is an affine combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$, we have that $M \subseteq \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$. WTS $\text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n) \subseteq M$. Fix $\mathbf{x} \in \text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n)$. By (ii), we have that $\mathbf{x}_1, \dots, \mathbf{x}_n \in M$, and by Theorem 2.1, we know that M is closed under affine combinations. Since $\mathbf{x}$ is an affine combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$, we deduce that $\mathbf{x} \in M$. This proves that $\text{Aff}(\mathbf{x}_1, \dots, \mathbf{x}_n) \subseteq M$. Thus, (i) holds. Q.E.D.

(c) Affine frames and affine bases

(c) Affine frames and affine bases

- We have extensively studied bases of (finite-dimensional) vector spaces.

(c) Affine frames and affine bases

- We have extensively studied bases of (finite-dimensional) vector spaces.
- For affine subspaces, we have two analogues of bases: "affine frames" and "affine bases."

(c) Affine frames and affine bases

- We have extensively studied bases of (finite-dimensional) vector spaces.
- For affine subspaces, we have two analogues of bases: "affine frames" and "affine bases."
- We first discuss affine frames, and then we discuss affine bases.

(c) Affine frames and affine bases

- We have extensively studied bases of (finite-dimensional) vector spaces.
- For affine subspaces, we have two analogues of bases: “affine frames” and “affine bases.”
- We first discuss affine frames, and then we discuss affine bases.
- As we shall see, the two concepts are closely related.

Definition

Let n be a non-negative integer, and let M be an n-dimensional affine subspace of a vector space V over a field $\mathbb{F}$. An *affine frame* of M is an ordered $(n+1)$-tuple $(\mathbf{a}, \mathbf{u}_1, \ldots, \mathbf{u}_n)$ of vectors of V s.t. M can be written in the form $M = \mathbf{a} + U$, where U is a linear subspace of V, and $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis of U.

Definition

Let n be a non-negative integer, and let M be an n-dimensional affine subspace of a vector space V over a field $\mathbb{F}$. An *affine frame* of M is an ordered $(n+1)$-tuple $(\mathbf{a}, \mathbf{u}_1, \dots, \mathbf{u}_n)$ of vectors of V s.t. M can be written in the form $M = \mathbf{a} + U$, where U is a linear subspace of V, and $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is a basis of U.

- **Remark:** Infinite-dimensional affine subspaces do not have affine frames.

Definition

Let n be a non-negative integer, and let M be an n-dimensional affine subspace of a vector space V over a field $\mathbb{F}$. An *affine frame* of M is an ordered $(n+1)$-tuple $(\mathbf{a}, \mathbf{u}_1, \ldots, \mathbf{u}_n)$ of vectors of V s.t. M can be written in the form $M = \mathbf{a} + U$, where U is a linear subspace of V, and $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis of U.

- **Remark:** Infinite-dimensional affine subspaces do not have affine frames.

- By Theorem 1.3 of Lecture Notes 7, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of a vector space V over a field $\mathbb{F}$, then every vector in V can be written as a linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in a unique way.

Definition

Let n be a non-negative integer, and let M be an n-dimensional affine subspace of a vector space V over a field $\mathbb{F}$. An *affine frame* of M is an ordered $(n+1)$-tuple $(\mathbf{a}, \mathbf{u}_1, \dots, \mathbf{u}_n)$ of vectors of V s.t. M can be written in the form $M = \mathbf{a} + U$, where U is a linear subspace of V, and $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is a basis of U.

- **Remark:** Infinite-dimensional affine subspaces do not have affine frames.

- By Theorem 1.3 of Lecture Notes 7, if $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of a vector space V over a field $\mathbb{F}$, then every vector in V can be written as a linear combination of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in a unique way.
- Our next theorem (next slide) is an analogue of this result for affine subspaces and affine frames.

Theorem 3.1

Let M be an affine subspace of a vector space V over a field $\mathbb{F}$, and let $(\mathbf{a}, \mathbf{u}_1, \ldots, \mathbf{u}_n)$ be an affine frame of M. Then for all $\mathbf{x} \in M$, there exist unique scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{x} = \mathbf{a} + \alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n$.

Proof (outline).

Theorem 3.1

Let M be an affine subspace of a vector space V over a field $\mathbb{F}$, and let $(\mathbf{a}, \mathbf{u}_1, \ldots, \mathbf{u}_n)$ be an affine frame of M. Then for all $\mathbf{x} \in M$, there exist unique scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{x} = \mathbf{a} + \alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n$.

Proof (outline). Set $U := \text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_n)$, so that $M = \mathbf{a} + U$.

Theorem 3.1

Let M be an affine subspace of a vector space V over a field $\mathbb{F}$, and let $(\mathbf{a}, \mathbf{u}_1, \dots, \mathbf{u}_n)$ be an affine frame of M. Then for all $\mathbf{x} \in M$, there exist unique scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{x} = \mathbf{a} + \alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n$.

Proof (outline). Set $U := \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_n)$, so that $M = \mathbf{a} + U$. Now for every vector $\mathbf{x} \in M$, there exists a unique vector $\mathbf{u} \in U$ s.t. $\mathbf{x} = \mathbf{a} + \mathbf{u}$. But for every vectror $\mathbf{u} \in U$, there exist unique scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{u} = \alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n$. Q.E.D.

Definition

Given vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ in a vector space V over a field $\mathbb{F}$, we say that vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in V$ are *affinely independent*, or that the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is *affinely independent*, if for all $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_1 \mathbf{x}_1 + \cdots + \alpha_n \mathbf{x}_n = \mathbf{0}$ and $\alpha_1 + \cdots + \alpha_n = 0$, we have that $\alpha_1 = \cdots = \alpha_n = 0$.

Definition

Given vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ in a vector space V over a field $\mathbb{F}$, we say that vectors $\mathbf{x}_1, \dots, \mathbf{x}_n \in V$ are *affinely independent*, or that the set $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is *affinely independent*, if for all $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_1 \mathbf{x}_1 + \dots + \alpha_n \mathbf{x}_n = \mathbf{0}$ and $\alpha_1 + \dots + \alpha_n = 0$, we have that $\alpha_1 = \dots = \alpha_n = 0$.

Proposition 3.2

Let V be a vector space over a field $\mathbb{F}$, and let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ ($n \geq 0$) be vectors in V. Then the following are equivalent:

(i) $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ are affinely independent;

(ii) there exists some $i \in \{0, 1, \dots, n\}$ s.t. vectors $\mathbf{x}_0 - \mathbf{x}_i, \dots, \mathbf{x}_{i-1} - \mathbf{x}_i, \mathbf{x}_{i+1} - \mathbf{x}_i, \dots, \mathbf{x}_n - \mathbf{x}_i$ are linearly independent;

(iii) for all $i \in \{0, 1, \dots, n\}$, vectors $\mathbf{x}_0 - \mathbf{x}_i, \dots, \mathbf{x}_{i-1} - \mathbf{x}_i, \mathbf{x}_{i+1} - \mathbf{x}_i, \dots, \mathbf{x}_n - \mathbf{x}_i$ are linearly independent.

Proof.

Proof. Obviously, (iii) implies (ii).

Proof. Obviously, (iii) implies (ii). We will show that (ii) implies (i), and that (i) implies (iii).

Proof. Obviously, (iii) implies (ii). We will show that (ii) implies (i), and that (i) implies (iii).

Suppose that (ii) holds. Let us prove (i).

Proof. Obviously, (iii) implies (ii). We will show that (ii) implies (i), and that (i) implies (iii).

Suppose that (ii) holds. Let us prove (i). By (ii) and by symmetry, WMA $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent.

Proof. Obviously, (iii) implies (ii). We will show that (ii) implies (i), and that (i) implies (iii).

Suppose that (ii) holds. Let us prove (i). By (ii) and by symmetry, WMA $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent. Now, fix scalars $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_0 \mathbf{x}_0 + \alpha_1 \mathbf{x}_1 + \dots + \alpha_n \mathbf{x}_n = \mathbf{0}$ and $\alpha_0 + \alpha_1 + \dots + \alpha_n = 0$. WTS $\alpha_0 = \alpha_1 = \dots = \alpha_n = 0$.

Proof. Obviously, (iii) implies (ii). We will show that (ii) implies (i), and that (i) implies (iii).

Suppose that (ii) holds. Let us prove (i). By (ii) and by symmetry, WMA $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent. Now, fix scalars $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_0\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n = \mathbf{0}$ and $\alpha_0 + \alpha_1 + \dots + \alpha_n = 0$. WTS $\alpha_0 = \alpha_1 = \dots = \alpha_n = 0$. Since $\alpha_0 + \alpha_1 + \dots + \alpha_n = 0$, we have that $\alpha_0 = -\alpha_1 - \dots - \alpha_n$,

Proof. Obviously, (iii) implies (ii). We will show that (ii) implies (i), and that (i) implies (iii).

Suppose that (ii) holds. Let us prove (i). By (ii) and by symmetry, WMA $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent. Now, fix scalars $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_0\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n = \mathbf{0}$ and $\alpha_0 + \alpha_1 + \dots + \alpha_n = 0$. WTS $\alpha_0 = \alpha_1 = \dots = \alpha_n = 0$. Since $\alpha_0 + \alpha_1 + \dots + \alpha_n = 0$, we have that $\alpha_0 = -\alpha_1 - \dots - \alpha_n$, and so

$$\begin{aligned} \mathbf{0} &= \alpha_0\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n \\ &= (-\alpha_1 - \dots - \alpha_n)\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n \\ &= \alpha_1(\mathbf{x}_1 - \mathbf{x}_0) + \dots + \alpha_n(\mathbf{x}_n - \mathbf{x}_0). \end{aligned}$$

Proof. Obviously, (iii) implies (ii). We will show that (ii) implies (i), and that (i) implies (iii).

Suppose that (ii) holds. Let us prove (i). By (ii) and by symmetry, WMA $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent. Now, fix scalars $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_0 \mathbf{x}_0 + \alpha_1 \mathbf{x}_1 + \dots + \alpha_n \mathbf{x}_n = \mathbf{0}$ and $\alpha_0 + \alpha_1 + \dots + \alpha_n = 0$. WTS $\alpha_0 = \alpha_1 = \dots = \alpha_n = 0$. Since $\alpha_0 + \alpha_1 + \dots + \alpha_n = 0$, we have that $\alpha_0 = -\alpha_1 - \dots - \alpha_n$, and so

$$\begin{aligned}
\mathbf{0} &= \alpha_0 \mathbf{x}_0 + \alpha_1 \mathbf{x}_1 + \dots + \alpha_n \mathbf{x}_n \\
&= (-\alpha_1 - \dots - \alpha_n)\mathbf{x}_0 + \alpha_1 \mathbf{x}_1 + \dots + \alpha_n \mathbf{x}_n \\
&= \alpha_1(\mathbf{x}_1 - \mathbf{x}_0) + \dots + \alpha_n(\mathbf{x}_n - \mathbf{x}_0).
\end{aligned}$$

Since vectors $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent, we see that $\alpha_1 = \dots = \alpha_n = 0$.

Proof. Obviously, (iii) implies (ii). We will show that (ii) implies (i), and that (i) implies (iii).

Suppose that (ii) holds. Let us prove (i). By (ii) and by symmetry, WMA $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent. Now, fix scalars $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_0\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n = \mathbf{0}$ and $\alpha_0 + \alpha_1 + \dots + \alpha_n = 0$. WTS $\alpha_0 = \alpha_1 = \dots = \alpha_n = 0$. Since $\alpha_0 + \alpha_1 + \dots + \alpha_n = 0$, we have that $\alpha_0 = -\alpha_1 - \dots - \alpha_n$, and so

$$\begin{aligned} \mathbf{0} &= \alpha_0\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n \\ &= (-\alpha_1 - \dots - \alpha_n)\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n \\ &= \alpha_1(\mathbf{x}_1 - \mathbf{x}_0) + \dots + \alpha_n(\mathbf{x}_n - \mathbf{x}_0). \end{aligned}$$

Since vectors $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent, we see that $\alpha_1 = \dots = \alpha_n = 0$. Since $\alpha_0 = -\alpha_1 - \dots - \alpha_n$, it follows that $\alpha_0 = 0$. This proves (i).

Proof (continued). Suppose now that (i) holds. Let us prove (iii).

Proof (continued). Suppose now that (i) holds. Let us prove (iii). By symmetry, it suffices to show that $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent.

Proof (continued). Suppose now that (i) holds. Let us prove (iii). By symmetry, it suffices to show that $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent. Fix scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_1(\mathbf{x}_1 - \mathbf{x}_0) + \dots + \alpha_n(\mathbf{x}_n - \mathbf{x}_0) = \mathbf{0}$. Then

$$(\underbrace{-\alpha_1 - \dots - \alpha_n}_{:=\alpha_0})\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n \quad = \quad \mathbf{0}.$$

Proof (continued). Suppose now that (i) holds. Let us prove (iii). By symmetry, it suffices to show that $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$ are linearly independent. Fix scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\alpha_1(\mathbf{x}_1 - \mathbf{x}_0) + \dots + \alpha_n(\mathbf{x}_n - \mathbf{x}_0) = \mathbf{0}$. Then

$$(\underbrace{-\alpha_1 - \dots - \alpha_n}_{:=\alpha_0})\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n = \mathbf{0}.$$

Since $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ are affinely independent, we now get that $\alpha_0 = \alpha_1 = \dots = \alpha_n = 0$, and we deduce that (iii) holds. Q.E.D.

Proposition 3.2

Let V be a vector space over a field $\mathbb{F}$, and let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ ($n \geq 0$) be vectors in V. Then the following are equivalent:

(i) $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ are affinely independent;

(ii) there exists some $i \in \{0, 1, \dots, n\}$ s.t. vectors $\mathbf{x}_0 - \mathbf{x}_i, \dots, \mathbf{x}_{i-1} - \mathbf{x}_i, \mathbf{x}_{i+1} - \mathbf{x}_i, \dots, \mathbf{x}_n - \mathbf{x}_i$ are linearly independent;

(iii) for all $i \in \{0, 1, \dots, n\}$, vectors $\mathbf{x}_0 - \mathbf{x}_i, \dots, \mathbf{x}_{i-1} - \mathbf{x}_i, \mathbf{x}_{i+1} - \mathbf{x}_i, \dots, \mathbf{x}_n - \mathbf{x}_i$ are linearly independent.

Definition

Let M be an affine subspace of a vector space V over a field $\mathbb{F}$. An *affine basis* (also called a *barycentric frame*) of M is a non-empty ordered set $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ of vectors in M s.t.

- vectors $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ are affinely independent;
- $M = \mathrm{Aff}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$.

Definition

Let M be an affine subspace of a vector space V over a field $\mathbb{F}$. An *affine basis* (also called a *barycentric frame*) of M is a non-empty ordered set $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of vectors in M s.t.

- vectors $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$ are affinely independent;
- $M = \mathrm{Aff}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$.

- By Theorem 3.4 (later!), every affine basis of an n-dimensional affine subspace contains exactly $n+1$ vectors.

Definition

Let M be an affine subspace of a vector space V over a field $\mathbb{F}$. An *affine basis* (also called a *barycentric frame*) of M is a non-empty ordered set $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ of vectors in M s.t.

- vectors $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ are affinely independent;
- $M = \text{Aff}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$.

- By Theorem 3.4 (later!), every affine basis of an n-dimensional affine subspace contains exactly $n+1$ vectors.
- **Remark:** Suppose that M is an affine subspace of a vector space V over a field $\mathbb{F}$, and let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ be vectors in V. In view of Corollary 2.3, we have that $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ is an affine basis of M iff all the following hold:
  1. $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in M$;
  2. vectors $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ are affinely independent;
  3. every vector in M can be expressed as an affine combination of $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$.

Theorem 3.3

Let M be an affine subspace of a vector space V over a field $\mathbb{F}$, and let $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be an affine basis of M. Then for all $\mathbf{x} \in M$, there exist unique scalars $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{F}$, called the *barycentric coordinates* of $\mathbf{x}$ with respect to the affine basis $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$, s.t. $\mathbf{x} = \sum_{i=0}^{n} \alpha_i \mathbf{x}_i$ and $\sum_{i=0}^{n} \alpha_i = 1$.

Proof.

Theorem 3.3

Let M be an affine subspace of a vector space V over a field $\mathbb{F}$, and let $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be an affine basis of M. Then for all $\mathbf{x} \in M$, there exist unique scalars $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{F}$, called the *barycentric coordinates* of $\mathbf{x}$ with respect to the affine basis $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$, s.t. $\mathbf{x} = \sum_{i=0}^{n} \alpha_i \mathbf{x}_i$ and $\sum_{i=0}^{n} \alpha_i = 1$.

Proof. Fix $\mathbf{x} \in M$. The existence of scalars $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{x} = \sum_{i=0}^{n} \alpha_i \mathbf{x}_i$ and $\sum_{i=0}^{n} \alpha_i = 1$ follows from the fact that $M = \mathrm{Aff}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$.

Theorem 3.3

Let M be an affine subspace of a vector space V over a field $\mathbb{F}$, and let $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be an affine basis of M. Then for all $\mathbf{x} \in M$, there exist unique scalars $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{F}$, called the *barycentric coordinates* of $\mathbf{x}$ with respect to the affine basis $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$, s.t. $\mathbf{x} = \sum_{i=0}^{n} \alpha_i \mathbf{x}_i$ and $\sum_{i=0}^{n} \alpha_i = 1$.

Proof. Fix $\mathbf{x} \in M$. The existence of scalars $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{x} = \sum_{i=0}^{n} \alpha_i \mathbf{x}_i$ and $\sum_{i=0}^{n} \alpha_i = 1$ follows from the fact that $M = \text{Aff}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$.

It remains to prove uniqueness.

Theorem 3.3

Let M be an affine subspace of a vector space V over a field $\mathbb{F}$, and let $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ be an affine basis of M. Then for all $\mathbf{x} \in M$, there exist unique scalars $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{F}$, called the *barycentric coordinates* of $\mathbf{x}$ with respect to the affine basis $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$, s.t. $\mathbf{x} = \sum_{i=0}^n \alpha_i \mathbf{x}_i$ and $\sum_{i=0}^n \alpha_i = 1$.

Proof. Fix $\mathbf{x} \in M$. The existence of scalars $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{x} = \sum_{i=0}^n \alpha_i \mathbf{x}_i$ and $\sum_{i=0}^n \alpha_i = 1$ follows from the fact that $M = \text{Aff}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$.

It remains to prove uniqueness. So, fix $\alpha_0, \alpha_1, \dots, \alpha_n, \beta_0, \beta_1, \dots, \beta_n \in \mathbb{F}$ s.t.

- $\mathbf{x} = \sum_{i=0}^n \alpha_i \mathbf{x}_i$ and $\sum_{i=0}^n \alpha_i = 1$;
- $\mathbf{x} = \sum_{i=0}^n \beta_i \mathbf{x}_i$ and $\sum_{i=0}^n \beta_i = 1$.

Then $\sum_{i=0}^n \alpha_i \mathbf{x}_i = \sum_{i=0}^n \beta_i \mathbf{x}_i$, and so $\sum_{i=0}^n (\alpha_i - \beta_i) \mathbf{x}_i = \mathbf{0}$.

Theorem 3.3

Let M be an affine subspace of a vector space V over a field $\mathbb{F}$, and let $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ be an affine basis of M. Then for all $\mathbf{x} \in M$, there exist unique scalars $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{F}$, called the *barycentric coordinates* of $\mathbf{x}$ with respect to the affine basis $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$, s.t. $\mathbf{x} = \sum_{i=0}^n \alpha_i \mathbf{x}_i$ and $\sum_{i=0}^n \alpha_i = 1$.

Proof. Fix $\mathbf{x} \in M$. The existence of scalars $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{x} = \sum_{i=0}^n \alpha_i \mathbf{x}_i$ and $\sum_{i=0}^n \alpha_i = 1$ follows from the fact that $M = \text{Aff}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$.

It remains to prove uniqueness. So, fix $\alpha_0, \alpha_1, \dots, \alpha_n, \beta_0, \beta_1, \dots, \beta_n \in \mathbb{F}$ s.t.

- $\mathbf{x} = \sum_{i=0}^n \alpha_i \mathbf{x}_i$ and $\sum_{i=0}^n \alpha_i = 1$;
- $\mathbf{x} = \sum_{i=0}^n \beta_i \mathbf{x}_i$ and $\sum_{i=0}^n \beta_i = 1$.

Then $\sum_{i=0}^n \alpha_i \mathbf{x}_i = \sum_{i=0}^n \beta_i \mathbf{x}_i$, and so $\sum_{i=0}^n (\alpha_i - \beta_i)\mathbf{x}_i = \mathbf{0}$. Also, $\sum_{i=0}^n (\alpha_i - \beta_i) = (\sum_{i=0}^n \alpha_i) - (\sum_{i=0}^n \beta_i) = 1 - 1 = 0$. Since vectors $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ are affinely independent, we deduce that $\alpha_i - \beta_i = 0$ (and consequently, $\alpha_i = \beta_i$) for all $i \in \{0, 1, \dots, n\}$. Q.E.D.

Theorem 3.1

Let M be an affine subspace of a vector space V over a field $\mathbb{F}$, and let $(\mathbf{a}, \mathbf{u}_1, \ldots, \mathbf{u}_n)$ be an affine frame of M. Then for all $\mathbf{x} \in M$, there exist unique scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{x} = \mathbf{a} + \alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n$.

Theorem 3.3

Let M be an affine subspace of a vector space V over a field $\mathbb{F}$, and let $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be an affine basis of M. Then for all $\mathbf{x} \in M$, there exist unique scalars $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{F}$, called the *barycentric coordinates* of $\mathbf{x}$ with respect to the affine basis $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$, s.t. $\mathbf{x} = \sum_{i=0}^{n} \alpha_i \mathbf{x}_i$ and $\sum_{i=0}^{n} \alpha_i = 1$.

- Theorem 3.4 (below) gives a relationship between affine bases and affine frames.

Theorem 3.4

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n \in V$. Then the following are equivalent:

(i) $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is an affine basis of M;

(ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

- Theorem 3.4 (below) gives a relationship between affine bases and affine frames.

Theorem 3.4

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n \in V$. Then the following are equivalent:

(i) $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is an affine basis of M;

(ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

- **Remark:** Since every affine frame of an n-dimensional affine subspace contains $n + 1$ vectors, Theorem 3.4 implies that every affine basis of an n-dimensional affine subspace contains exactly $n + 1$ vectors.

Theorem 3.4

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n \in V$. Then the following are equivalent:

(i) $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is an affine basis of M;

(ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

Proof.

Theorem 3.4

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in V$. Then the following are equivalent:

1. (i) $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ is an affine basis of M;
2. (ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

Proof. We prove "(i) $\implies$ (ii)." The proof of "(ii) $\implies$ (i)" is similar (see the Lecture Notes).

Theorem 3.4

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in V$. Then the following are equivalent:

(i) $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ is an affine basis of M;

(ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

Proof. We prove "(i) $\implies$ (ii)." The proof of "(ii) $\implies$ (i)" is similar (see the Lecture Notes).

Assume (i) holds. WTS (ii).

Theorem 3.4

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in V$. Then the following are equivalent:

(i) $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ is an affine basis of M;

(ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

Proof. We prove "(i) $\Longrightarrow$ (ii)." The proof of "(ii) $\Longrightarrow$ (i)" is similar (see the Lecture Notes).

Assume (i) holds. WTS (ii). Since (i) holds, we know that $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in M$.

Theorem 3.4

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in V$. Then the following are equivalent:

(i) $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ is an affine basis of M;

(ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

Proof. We prove "(i) $\implies$ (ii)." The proof of "(ii) $\implies$ (i)" is similar (see the Lecture Notes).

Assume (i) holds. WTS (ii). Since (i) holds, we know that $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in M$. So, by Theorem 1.1(b), M can be written in the form $M = \mathbf{x}_0 + U$ for some linear subspace U of V.

Theorem 3.4

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in V$. Then the following are equivalent:

1. (i) $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ is an affine basis of M;
2. (ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

Proof. We prove "(i) $\implies$ (ii)." The proof of "(ii) $\implies$ (i)" is similar (see the Lecture Notes).

Assume (i) holds. WTS (ii). Since (i) holds, we know that $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in M$. So, by Theorem 1.1(b), M can be written in the form $M = \mathbf{x}_0 + U$ for some linear subspace U of V. It remains to show that $\{\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0\}$ is a basis of U.

Theorem 3.4

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in V$. Then the following are equivalent:

(i) $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ is an affine basis of M;

(ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

Proof. We prove "(i) $\Longrightarrow$ (ii)." The proof of "(ii) $\Longrightarrow$ (i)" is similar (see the Lecture Notes).

Assume (i) holds. WTS (ii). Since (i) holds, we know that $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in M$. So, by Theorem 1.1(b), M can be written in the form $M = \mathbf{x}_0 + U$ for some linear subspace U of V. It remains to show that $\{\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0\}$ is a basis of U. Since $\mathbf{x}_1, \dots, \mathbf{x}_n \in M = \mathbf{x}_0 + U$, it is clear that $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0 \in U$.

Theorem 3.4

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in V$. Then the following are equivalent:

1. (i) $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ is an affine basis of M;
2. (ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

Proof. We prove "(i) $\implies$ (ii)." The proof of "(ii) $\implies$ (i)" is similar (see the Lecture Notes).

Assume (i) holds. WTS (ii). Since (i) holds, we know that $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in M$. So, by Theorem 1.1(b), M can be written in the form $M = \mathbf{x}_0 + U$ for some linear subspace U of V. It remains to show that $\{\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0\}$ is a basis of U. Since $\mathbf{x}_1, \dots, \mathbf{x}_n \in M = \mathbf{x}_0 + U$, it is clear that $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0 \in U$. Moreover, since the set $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ is affinely independent (by (i)), Proposition 3.2 guarantees that $\{\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0\}$ is linearly independent.

Theorem 3.4

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n \in V$. Then the following are equivalent:

1. (i) $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is an affine basis of M;
2. (ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

Proof (continued). It remains to show that any vector in U can be written as a linear combination of $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0$.

Theorem 3.4

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in V$. Then the following are equivalent:

1. (i) $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ is an affine basis of M;
2. (ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

Proof (continued). It remains to show that any vector in U can be written as a linear combination of $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$. Fix any $\mathbf{u} \in U$.

Theorem 3.4

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n \in V$. Then the following are equivalent:

1. (i) $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is an affine basis of M;
2. (ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

Proof (continued). It remains to show that any vector in U can be written as a linear combination of $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0$. Fix any $\mathbf{u} \in U$. Then $\mathbf{x} := \mathbf{x}_0 + \mathbf{u} \in \mathbf{x}_0 + U = M$,

Theorem 3.4

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in V$. Then the following are equivalent:

1. (i) $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ is an affine basis of M;
2. (ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

Proof (continued). It remains to show that any vector in U can be written as a linear combination of $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$. Fix any $\mathbf{u} \in U$. Then $\mathbf{x} := \mathbf{x}_0 + \mathbf{u} \in \mathbf{x}_0 + U = M$, and so by (i), $\exists \alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{x} = \alpha_0\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \cdots + \alpha_n\mathbf{x}_n$ and $\alpha_0 + \alpha_1 + \cdots + \alpha_n = 1$.

Theorem 3.4

Let V be a vector space over a field $\mathbb{F}$, let M be an affine subspace of V, and let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n \in V$. Then the following are equivalent:

1. (i) $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ is an affine basis of M;
2. (ii) $(\mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0)$ is an affine frame of M.

Proof (continued). It remains to show that any vector in U can be written as a linear combination of $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$. Fix any $\mathbf{u} \in U$. Then $\mathbf{x} := \mathbf{x}_0 + \mathbf{u} \in \mathbf{x}_0 + U = M$, and so by (i), $\exists \alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $\mathbf{x} = \alpha_0\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n$ and $\alpha_0 + \alpha_1 + \dots + \alpha_n = 1$. We now compute:

$$\begin{aligned} \mathbf{u} &= \mathbf{x} - \mathbf{x}_0 \\ &= \underbrace{(\alpha_0\mathbf{x}_0 + \alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n)}_{=\mathbf{x}} - \underbrace{(\alpha_0 + \alpha_1 + \dots + \alpha_n)}_{=1}\mathbf{x}_0 \\ &= \alpha_1(\mathbf{x}_1 - \mathbf{x}_0) + \dots + \alpha_n(\mathbf{x}_n - \mathbf{x}_0). \end{aligned}$$

So, $\mathbf{u}$ is a linear combination of $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_n - \mathbf{x}_0$. Q.E.D.

(d) Affine transofrmations

(d) Affine transofrmations

Definition

Suppose that V_1 and V_2 are vector spaces over a field $\mathbb{F}$. A function $f : V_1 \to V_2$ is called an *affine transformation* (or an *affine function*) if there exists a linear transformation $g : V_1 \to V_2$ and a vector $\mathbf{b} \in V_2$ s.t. for all $\mathbf{x} \in V_1$, we have that $f(\mathbf{x}) = g(\mathbf{x}) + \mathbf{b}$.

(d) Affine transofrmations

Definition

Suppose that V_1 and V_2 are vector spaces over a field $\mathbb{F}$. A function $f : V_1 \to V_2$ is called an *affine transformation* (or an *affine function*) if there exists a linear transformation $g : V_1 \to V_2$ and a vector $\mathbf{b} \in V_2$ s.t. for all $\mathbf{x} \in V_1$, we have that $f(\mathbf{x}) = g(\mathbf{x}) + \mathbf{b}$.

- Obviously, every linear transformation f is affine (we simply take $g := f$ and $\mathbf{b} := \mathbf{0}$).

(d) Affine transofrmations

Definition

Suppose that V_1 and V_2 are vector spaces over a field $\mathbb{F}$. A function $f : V_1 \to V_2$ is called an *affine transformation* (or an *affine function*) if there exists a linear transformation $g : V_1 \to V_2$ and a vector $\mathbf{b} \in V_2$ s.t. for all $\mathbf{x} \in V_1$, we have that $f(\mathbf{x}) = g(\mathbf{x}) + \mathbf{b}$.

- Obviously, every linear transformation f is affine (we simply take $g := f$ and $\mathbf{b} := \mathbf{0}$).
- We now state a few propositions/theorems about affine transformations.
  - The proofs are in the Lecture Notes (here, we omit them).

Proposition 4.1

Let V_1 and V_2 be vector spaces over a field $\mathbb{F}$, and let $f : V_1 \to V_2$ be a linear transformation. Then f is linear iff $f(\mathbf{0}) = \mathbf{0}$.

Proposition 4.1

Let V_1 and V_2 be vector spaces over a field $\mathbb{F}$, and let $f : V_1 \to V_2$ be a linear transformation. Then f is linear iff $f(\mathbf{0}) = \mathbf{0}$.

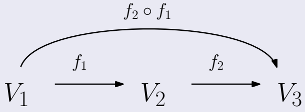
Theorem 4.2

Let V_1, V_2, V_3 be vector spaces over a field $\mathbb{F}$. Then all the following hold:

(a) for all affine transformations $f_1, f_2 : V_1 \to V_2$, we have that $f_1 + f_2$ is an affine transformation;

(b) for all affine transformations $f : V_1 \to V_2$ and scalars α, we have that αf is an affine transformation;

(c) for all affine transformations $f_1 : V_1 \to V_2$ and $f_2 : V_2 \to V_3$, we have that $f_2 \circ f_1$ is an affine transformation.



Theorem 4.3

Let V_1 and V_2 be vector spaces over a field $\mathbb{F}$, let $g : V_1 \to V_2$ be a linear transformation, and let U_2 be a linear subspace of V_2. Then $g^{-1}[U_2]$ is a linear subspace of V_1.

Theorem 4.3

Let V_1 and V_2 be vector spaces over a field $\mathbb{F}$, let $g : V_1 \to V_2$ be a linear transformation, and let U_2 be a linear subspace of V_2. Then $g^{-1}[U_2]$ is a linear subspace of V_1.

Theorem 4.4

Let V_1 and V_2 be vector spaces over a field $\mathbb{F}$, and let $f : V_1 \to V_2$ be an affine transformation. Then all the following hold:

(a) for every affine subspace M_1 of V_1, we have that $f[M_1]$ is an affine subspace of V_2;

(b) $\text{Im}(f)$ is an affine subspace of V_2;

(c) for every affine subspace M_2 of V_2, $f^{-1}[M_2]$ is either empty or an affine subspace of V_1;

(d) for every $\mathbf{b} \in V$, the set of solutions of the equation $f(\mathbf{x}) = \mathbf{b}$ is either empty or an affine subspace of V_1.

Theorem 4.4

Let V_1 and V_2 be vector spaces over a field $\mathbb{F}$, and let $f : V_1 \to V_2$ be an affine transformation. Then all the following hold:

(a) for every affine subspace M_1 of V_1, we have that $f[M_1]$ is an affine subspace of V_2;

(b) $\text{Im}(f)$ is an affine subspace of V_2;

(c) for every affine subspace M_2 of V_2, $f^{-1}[M_2]$ is either empty or an affine subspace of V_1;

(d) for every $\mathbf{b} \in V$, the set of solutions of the equation $f(\mathbf{x}) = \mathbf{b}$ is either empty or an affine subspace of V_1.

Theorem 4.4

Let V_1 and V_2 be vector spaces over a field $\mathbb{F}$, and let $f : V_1 \to V_2$ be an affine transformation. Then all the following hold:

- (a) for every affine subspace M_1 of V_1, we have that $f[M_1]$ is an affine subspace of V_2;
- (b) $\text{Im}(f)$ is an affine subspace of V_2;
- (c) for every affine subspace M_2 of V_2, $f^{-1}[M_2]$ is either empty or an affine subspace of V_1;
- (d) for every $\mathbf{b} \in V$, the set of solutions of the equation $f(\mathbf{x}) = \mathbf{b}$ is either empty or an affine subspace of V_1.

Corollary 4.5

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times m}$ and $\mathbf{b} \in \mathbb{F}^n$. Then the solution set of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is either empty or an affine subspace of $\mathbb{F}^m$.