Linear Algebra 2

Lecture #21
Positive (semi-)definite matrices

Irena Penev

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- Throughout this lecture, $\cdot$ is the standard scalar product in $\mathbb{R}^n$, and $||\cdot||$ is the induced norm.

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A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is said to be

- *positive semi-definite* if $\mathbf{x}^T A \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$;
- *positive definite* if $\mathbf{x}^T A \mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

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- Obviously, every positive definite matrix is positive semi-definite.

Example 1.1

The identity matrix I_n is positive definite. This is because I_n is symmetric, and $\forall \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$: $\mathbf{x}^T I_n\mathbf{x} = \mathbf{x}^T\mathbf{x} = \mathbf{x} \cdot \mathbf{x} > 0$.

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- However, note that for any $A \in \mathbb{R}^{n\times n}$, the matrix $\frac{1}{2}(A + A^T)$ is symmetric, and for all vectors $\mathbf{x} \in \mathbb{R}^n$, we have that

$$\begin{aligned}
\mathbf{x}^T\Big(\tfrac{1}{2}(A + A^T)\Big)\mathbf{x} &= \tfrac{1}{2}(\mathbf{x}^T A\mathbf{x}) + \tfrac{1}{2}(\mathbf{x}^T A^T\mathbf{x}) \\
&\overset{(*)}{=} \tfrac{1}{2}(\mathbf{x}^T A\mathbf{x}) + \tfrac{1}{2}(\mathbf{x}^T A^T\mathbf{x})^T \\
&= \tfrac{1}{2}(\mathbf{x}^T A\mathbf{x}) + \tfrac{1}{2}(\mathbf{x}^T A\mathbf{x}) \\
&= \mathbf{x}^T A\mathbf{x},
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where (*) follows from the fact that $\mathbf{x}^T A\mathbf{x}$ is a 1×1 matrix, and is consequently symmetric.

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- However, note that for any $A \in \mathbb{R}^{n \times n}$, the matrix $\frac{1}{2}(A + A^T)$ is symmetric, and for all vectors $\mathbf{x} \in \mathbb{R}^n$, we have that

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  where (*) follows from the fact that $\mathbf{x}^T A\mathbf{x}$ is a 1×1 matrix, and is consequently symmetric.
- So, instead of considering an arbitrary matrix A, we can consider the symmetric matrix $\frac{1}{2}(A + A^T)$ in this context.

2. Positive definite matrices and the scalar product

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Theorem 2.2

For any $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, the following are equivalent:

(i) $\langle \cdot, \cdot \rangle$ is a scalar product on $\mathbb{R}^n$;

(ii) there exists a positive definite matrix $A \in \mathbb{R}^{n \times n}$ s.t. for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A \mathbf{y}$.

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- We will prove Theorem 2.2. But first, let us recall the definition of a scalar product (from Lecture 11), and let us prove a proposition about products of the form $\mathbf{x}^T A\mathbf{y}$, where A is a square matrix and $\mathbf{x}, \mathbf{y}$ are vectors.

Definition

A *scalar product* (also called *inner product*) in a vector space V over the field $\mathbb{R}$ is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies the following axioms:

r.1. for all $\mathbf{x} \in V$, $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, and equality holds iff $\mathbf{x} = \mathbf{0}$;

r.2. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$;

r.3. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{R}$, $\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$;

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r.4. for all $\mathbf{x}, \mathbf{y} \in V$, $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.

We saw in Lecture 11 that these four axioms imply the following:

r.2'. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$;

r.3'. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{R}$, $\langle \mathbf{x}, \alpha\mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$.

Proposition 2.1

Let $\mathbb{F}$ be a field. Then for all matrices $A = [\ a_{i,j}\ ]_{n\times n}$ in $\mathbb{F}^{n\times n}$, and all vectors $\mathbf{x} = [\ x_1 \quad \dots \quad x_n\ ]^T$ and $\mathbf{y} = [\ y_1 \quad \dots \quad y_n\ ]^T$ in $\mathbb{F}^n$, we have that

$$\mathbf{x}^T A\mathbf{y} \quad = \quad \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}x_i y_j.$$

Proof.

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$$\mathbf{x}^T A\mathbf{y} \quad = \quad \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}x_i y_j.$$

Proof. First of all, we have that

$$A\mathbf{y} \quad = \quad \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad = \quad \begin{bmatrix} \sum_{j=1}^{n} a_{1,j}y_j \\ \sum_{j=1}^{n} a_{2,j}y_j \\ \vdots \\ \sum_{j=1}^{n} a_{n,j}y_j \end{bmatrix}.$$

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$$\mathbf{x}^T A\mathbf{y} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}x_i y_j.$$

Proof (continued). But now

$$\begin{aligned}
\mathbf{x}^T A\mathbf{y} &= \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} \sum_{j=1}^{n} a_{1,j}y_j \\ \sum_{j=1}^{n} a_{2,j}y_j \\ \vdots \\ \sum_{j=1}^{n} a_{n,j}y_j \end{bmatrix} \\
&= \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} a_{i,j}y_j \right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}x_i y_j
\end{aligned}$$

Q.E.D.

Theorem 2.2

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Proof. Suppose first that (i) holds.

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Proof. Suppose first that (i) holds. $\forall i, j \in \{1, \dots, n\}$, we set $a_{i,j} := \langle \mathbf{e}_i, \mathbf{e}_j\rangle$. Set $A := \begin{bmatrix} a_{i,j} \end{bmatrix}_{n\times n}$. Then for all vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$ in $\mathbb{R}^n$:

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{y}\rangle &= \langle \sum_{i=1}^n x_i\mathbf{e}_i, \sum_{j=1}^n y_j\mathbf{e}_j\rangle & \\
&= \sum_{i=1}^n \sum_{j=1}^n x_i y_i \langle \mathbf{e}_i, \mathbf{e}_j\rangle & \text{because } \langle\cdot,\cdot\rangle \text{ is a scalar product in } \mathbb{R}^n \\
&= \sum_{i=1}^n \sum_{j=1}^n x_i y_i a_{i,j} & \text{because } a_{i,j} = \langle \mathbf{e}_i, \mathbf{e}_j\rangle \\
&= \mathbf{x}^T A\mathbf{y} & \text{by Proposition 2.1.}
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Proof (continued). It remains to show that A is positive definite.

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Proof (continued). It remains to show that A is positive definite. First of all, A is symmetric because $\forall i, j \in \{1, \dots, n\}$, we have that

$$a_{i,j} \overset{(*)}{=} \langle \mathbf{e}_i, \mathbf{e}_j \rangle \overset{(**)}{=} \langle \mathbf{e}_j, \mathbf{e}_i \rangle \overset{(*)}{=} a_{j,i},$$

where both instances of (*) follow from the construction of A, and (**) follows from (i).

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where both instances of (*) follow from the construction of A, and (**) follows from (i). On the other hand, for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, we have that

$$\mathbf{x}^T A \mathbf{x} = \langle \mathbf{x}, \mathbf{x} \rangle \overset{(*)}{>} 0,$$

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where both instances of (*) follow from the construction of A, and (**) follows from (i). On the other hand, for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, we have that

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where (*) follows from (i). So, (ii) holds.

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Proof (continued). Suppose now that (ii) holds, and let the matrix A be as in (ii).

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Proof (continued). Suppose now that (ii) holds, and let the matrix A be as in (ii). We must prove (i). We verify the axioms of a scalar product one by one.

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Proof (continued). Suppose now that (ii) holds, and let the matrix A be as in (ii). We must prove (i). We verify the axioms of a scalar product one by one.

r.1. For every $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, we have that $\langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^T A \mathbf{x} \overset{(*)}{>} 0$, where (*) follows from the fact that A is positive definite.

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Proof (continued). Suppose now that (ii) holds, and let the matrix A be as in (ii). We must prove (i). We verify the axioms of a scalar product one by one.

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Proof (continued). Suppose now that (ii) holds, and let the matrix A be as in (ii). We must prove (i). We verify the axioms of a scalar product one by one.

r.1. For every $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, we have that $\langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^T A \mathbf{x} \overset{(*)}{>} 0$, where (*) follows from the fact that A is positive definite. On the other hand, we have that $\langle \mathbf{0}, \mathbf{0} \rangle = \mathbf{0}^T A \mathbf{0} = 0$. This proves that for all $\mathbf{x} \in \mathbb{R}^n$, we have that $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, and that equality holds iff $\mathbf{x} = \mathbf{0}$.

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Proof (continued). r.2. For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, we have that:

$$\begin{aligned}
\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle &= (\mathbf{x} + \mathbf{y})^T A \mathbf{z} \\
&= (\mathbf{x}^T + \mathbf{y}^T) A \mathbf{z} \\
&= \mathbf{x}^T A \mathbf{z} + \mathbf{y}^T A \mathbf{z} \\
&= \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle.
\end{aligned}$$

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Proof (continued). r.3. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we have that

$$\langle \alpha \mathbf{x}, \mathbf{y} \rangle = (\alpha \mathbf{x})^T A \mathbf{y} = \alpha(\mathbf{x}^T A \mathbf{y}) = \alpha \langle \mathbf{x}, \mathbf{y} \rangle.$$

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Proof (continued). r.3. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we have that

$$\langle \alpha\mathbf{x}, \mathbf{y} \rangle \;=\; (\alpha\mathbf{x})^T A\mathbf{y} \;=\; \alpha(\mathbf{x}^T A\mathbf{y}) \;=\; \alpha\langle \mathbf{x}, \mathbf{y} \rangle.$$

r.4. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have that

$$\langle \mathbf{x}, \mathbf{y} \rangle \;=\; \mathbf{x}^T A\mathbf{y} \;\overset{(*)}{=}\; (\mathbf{x}^T A\mathbf{y})^T \;=\; \mathbf{y}^T A^T \mathbf{x} \;\overset{(**)}{=}\; \mathbf{y}^T A\mathbf{x} \;=\; \langle \mathbf{y}, \mathbf{x} \rangle,$$

where in (*), we used the fact that A is a 1×1 (and consequently, symmetric) matrix, and in (**), we used the fact that A is symmetric.

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For any $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, the following are equivalent:

1. (i) $\langle \cdot, \cdot \rangle$ is a scalar product on $\mathbb{R}^n$;
2. (ii) there exists a positive definite matrix $A \in \mathbb{R}^{n \times n}$ s.t. for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A \mathbf{y}$.

Proof (continued). r.3. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we have that

$$\langle \alpha\mathbf{x}, \mathbf{y} \rangle = (\alpha\mathbf{x})^T A\mathbf{y} = \alpha(\mathbf{x}^T A\mathbf{y}) = \alpha\langle \mathbf{x}, \mathbf{y} \rangle.$$

r.4. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A\mathbf{y} \overset{(*)}{=} (\mathbf{x}^T A\mathbf{y})^T = \mathbf{y}^T A^T \mathbf{x} \overset{(**)}{=} \mathbf{y}^T A\mathbf{x} = \langle \mathbf{y}, \mathbf{x} \rangle,$$

where in (*), we used the fact that A is a 1×1 (and consequently, symmetric) matrix, and in (**), we used the fact that A is symmetric.

This proves that (i) holds. Q.E.D.

Theorem 2.2

For any $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, the following are equivalent:

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- **Remark:** Positive (semi-)definite matrices also play an important role in optimization, but we shall not discuss this in this course.

3. Basic properties of positive (semi-)definite matrices

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- Let us say that the main diagonal of a matrix $A = [\ a_{i,j}\ ]_{n\times n}$ in $\mathbb{R}^{n\times n}$ is *non-negative* (resp. *positive*) if $a_{1,1}, \ldots, a_{n,n} \geq 0$ (resp. $a_{1,1}, \ldots, a_{n,n} > 0$).
  - In other words, the main diagonal of a square matrix is non-negative (resp. positive) if all the entries on the main diagonal of that matrix are non-negative (resp. positive). The following proposition gives a necessary (but not sufficient) condition for positive (semi-)definiteness.

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Proposition 3.1

(a) The main diagonal of any positive semi-definite is non-negative.

(b) The main diagonal of any positive definite is positive.

Proof.

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    - In other words, the main diagonal of a square matrix is non-negative (resp. positive) if all the entries on the main diagonal of that matrix are non-negative (resp. positive). The following proposition gives a necessary (but not sufficient) condition for positive (semi-)definiteness.

Proposition 3.1

(a) The main diagonal of any positive semi-definite is non-negative.

(b) The main diagonal of any positive definite is positive.

Proof. Fix a matrix $A = [\ a_{i,j}\ ]_{n \times n}$ in $\mathbb{R}^{n \times n}$. Note that for all indices $i \in \{1, \dots, n\}$, we have that $\mathbf{e}_i^T A \mathbf{e}_i = a_{i,i}$. The result now follows from the definition of positive (semi-)definiteness.

Theorem 3.2

a. If $A, B \in \mathbb{R}^{n \times n}$ are both positive definite, then $A + B$ is positive definite.
b. If $A \in \mathbb{R}^{n \times n}$ is positive definite and $\alpha > 0$, then αA is positive definite.
c. If $A \in \mathbb{R}^{n \times n}$ is positive definite, then A is invertible and its inverse A^{-1} is positive definite.

Proof.

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Proof. (a) and (b) are trivial. Let us prove (c). Fix a positive definite matrix $A \in \mathbb{R}^{n\times n}$. We first prove that A is invertible.

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(a) If $A, B \in \mathbb{R}^{n \times n}$ are both positive definite, then $A + B$ is positive definite.

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Theorem 3.2

- If $A \in \mathbb{R}^{n \times n}$ is positive definite, then A is invertible and its inverse A^{-1} is positive definite.

Proof (continued). It remains to show that A^{-1} is positive definite.

Theorem 3.2

- (c) If $A \in \mathbb{R}^{n \times n}$ is positive definite, then A is invertible and its inverse A^{-1} is positive definite.

Proof (continued). It remains to show that A^{-1} is positive definite. Since A is positive definite, it is symmetric; consequently, A^{-1} is also symmetric.

Theorem 3.2

- (c) If $A \in \mathbb{R}^{n \times n}$ is positive definite, then A is invertible and its inverse A^{-1} is positive definite.

Proof (continued). It remains to show that A^{-1} is positive definite. Since A is positive definite, it is symmetric; consequently, A^{-1} is also symmetric. Now, fix any $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

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- (c) If $A \in \mathbb{R}^{n \times n}$ is positive definite, then A is invertible and its inverse A^{-1} is positive definite.

Proof (continued). It remains to show that A^{-1} is positive definite. Since A is positive definite, it is symmetric; consequently, A^{-1} is also symmetric. Now, fix any $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Since $\mathbf{x} \neq \mathbf{0}$ and A^{-1} is invertible, we see that $A^{-1}\mathbf{x} \neq 0$.

Theorem 3.2

(c) If $A \in \mathbb{R}^{n \times n}$ is positive definite, then A is invertible and its inverse A^{-1} is positive definite.

Proof (continued). It remains to show that A^{-1} is positive definite. Since A is positive definite, it is symmetric; consequently, A^{-1} is also symmetric. Now, fix any $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Since $\mathbf{x} \neq \mathbf{0}$ and A^{-1} is invertible, we see that $A^{-1}\mathbf{x} \neq 0$. But now we have the following:

$$\begin{aligned}
\mathbf{x}^T A^{-1}\mathbf{x} &= \mathbf{x}^T A^{-1} A A^{-1}\mathbf{x} \\
&= ((A^{-1})^T\mathbf{x})^T A(A^{-1}\mathbf{x}) \\
&= (A^{-1}\mathbf{x})^T A(A^{-1}\mathbf{x}) && \text{because } A^{-1} \text{ is symmetric} \\
&> 0 && \text{because } A \text{ is positive definite and } A^{-1}\mathbf{x} \neq 0.
\end{aligned}$$

So, A^{-1} is positive definite. Q.E.D.

Theorem 3.3

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Then the following are equivalent:

(i) A is positive definite;

(ii) all eigenvalues of A are positive;

(iii) there exists an invertible matrix $U \in \mathbb{R}^{n \times n}$ s.t. $A = U^T U$.

Proof.

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"(i) $\Longrightarrow$ (ii)": Assume that (i) holds.

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Proof. We will prove "(i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i)."

"(i) $\implies$ (ii)": Assume that (i) holds. Fix any eigenvalue λ of A, and let $\mathbf{x}$ be an associated eigenvector. After possibly normalizing $\mathbf{x}$ (i.e. replacing $\mathbf{x}$ by $\frac{\mathbf{x}}{||\mathbf{x}||}$), we may assume that $||\mathbf{x}|| = 1$.

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Proof. We will prove "(i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i)."

"(i) $\Longrightarrow$ (ii)": Assume that (i) holds. Fix any eigenvalue λ of A, and let $\mathbf{x}$ be an associated eigenvector. After possibly normalizing $\mathbf{x}$ (i.e. replacing $\mathbf{x}$ by $\frac{\mathbf{x}}{||\mathbf{x}||}$), we may assume that $||\mathbf{x}|| = 1$. Since A is positive definite, we have that $\mathbf{x}^T A\mathbf{x} > 0$. But note that

$$\mathbf{x}^T A\mathbf{x} \;=\; \mathbf{x}^T(\lambda\mathbf{x}) \;=\; \lambda(\mathbf{x}^T\mathbf{x}) \;=\; \lambda(\mathbf{x}\cdot\mathbf{x}) \;=\; \lambda||\mathbf{x}||^2 \;=\; \lambda.$$

So, $\lambda = \mathbf{x}^T A\mathbf{x} > 0$. Thus, (ii) holds.

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Proof (continued). "(ii) $\Longrightarrow$ (iii)": Assume that (ii) holds.

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Proof (continued). "(ii) $\Longrightarrow$ (iii)": Assume that (ii) holds. Since A is symmetric, it is orthogonally diagonalizable (by Theorem 3.5 of Lecture Notes 20).

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Proof (continued). "(ii) $\Longrightarrow$ (iii)": Assume that (ii) holds. Since A is symmetric, it is orthogonally diagonalizable (by Theorem 3.5 of Lecture Notes 20). Let $D = D(\lambda_1, \ldots, \lambda_n)$ be a diagonal and Q an orthogonal matrix, both in $\mathbb{R}^{n \times n}$, s.t. $D = Q^T AQ$, and consequently, $A = QDQ^T$.

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Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Then the following are equivalent:

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Proof (continued). "(ii) $\Longrightarrow$ (iii)": Assume that (ii) holds. Since A is symmetric, it is orthogonally diagonalizable (by Theorem 3.5 of Lecture Notes 20). Let $D = D(\lambda_1, \dots, \lambda_n)$ be a diagonal and Q an orthogonal matrix, both in $\mathbb{R}^{n \times n}$, s.t. $D = Q^T A Q$, and consequently, $A = QDQ^T$. Then $\lambda_1, \dots, \lambda_n$ are all eigenvalues of A, and so by (ii), $\lambda_1, \dots, \lambda_n > 0$. Now, set $\widetilde{D} := D(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$; clearly, $\widetilde{D}^2 = D$.

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Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then the following are equivalent:

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Proof (continued). "(ii) $\Longrightarrow$ (iii)": Assume that (ii) holds. Since A is symmetric, it is orthogonally diagonalizable (by Theorem 3.5 of Lecture Notes 20). Let $D = D(\lambda_1, \dots, \lambda_n)$ be a diagonal and Q an orthogonal matrix, both in $\mathbb{R}^{n\times n}$, s.t. $D = Q^T AQ$, and consequently, $A = QDQ^T$. Then $\lambda_1, \dots, \lambda_n$ are all eigenvalues of A, and so by (ii), $\lambda_1, \dots, \lambda_n > 0$. Now, set $\widetilde{D} := D(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$; clearly, $\widetilde{D}^2 = D$. Next, set $U := \widetilde{D}Q$. Since $\widetilde{D}$ and Q are both invertible, so is U. But now (next slide):

Theorem 3.3

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Then the following are equivalent:

1. A is positive definite;
2. all eigenvalues of A are positive;
3. there exists an invertible matrix $U \in \mathbb{R}^{n \times n}$ s.t. $A = U^T U$.

Proof (continued). "(ii) $\Longrightarrow$ (iii)" (continued):

$$\begin{aligned} U^T U &= (\widetilde{D}Q)^T(\widetilde{D}Q) \\ &= Q^T \widetilde{D}^T \widetilde{D} Q \\ &= Q^T \widetilde{D}^2 Q \\ &= Q^T D Q \\ &= A. \end{aligned}$$

So, (iii) holds.

Theorem 3.3

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Then the following are equivalent:

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3. (iii) there exists an invertible matrix $U \in \mathbb{R}^{n \times n}$ s.t. $A = U^T U$.

Proof (continued). "(iii) $\Longrightarrow$ (i)": Assume (iii), and fix an invertible matrix U s.t. $A = U^T U$. Fix any vector $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Then

$$\begin{array}{rcl} \mathbf{x}^T A \mathbf{x} & = & \mathbf{x}^T U^T U \mathbf{x} \\ & = & (U\mathbf{x})^T (U\mathbf{x}) \\ & = & (U\mathbf{x}) \cdot (U\mathbf{x}) \\ & = & \|U\mathbf{x}\|^2 \\ & \stackrel{(*)}{>} & 0, \end{array}$$

where (*) follows from the fact that $U\mathbf{x} \neq \mathbf{0}$ (because U is invertible and $\mathbf{x} \neq \mathbf{0}$). So, (i) holds. Q.E.D.

Theorem 3.3

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Then the following are equivalent:

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Theorem 3.4

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then the following are equivalent:

1. (i) A is positive semi-definite;
2. (ii) all eigenvalues of A are non-negative;
3. (iii) there exists a matrix $U \in \mathbb{R}^{n\times n}$ s.t. $A = U^T U$.

Proof. Analogous to the proof of Theorem 3.3.

Proposition 3.5

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix.

(a) If A is positive semi-definite, then $\det(A)$ and $\mathrm{trace}(A)$ are both non-negative.

(b) If A is positive definite, then $\det(A)$ and $\mathrm{trace}(A)$ are both positive.

Proof.

Proposition 3.5

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix.

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Proof. Since A is symmetric, Corollary 2.4 of Lecture Notes 20 guarantees that it has n real eigenvalues (with algebraic multiplicities taken into account). So, let $\{\lambda_1, \ldots, \lambda_n\}$ be the spectrum of A.

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Proof. Since A is symmetric, Corollary 2.4 of Lecture Notes 20 guarantees that it has n real eigenvalues (with algebraic multiplicities taken into account). So, let $\{\lambda_1, \dots, \lambda_n\}$ be the spectrum of A. By Theorem 2.11 of Lecture Notes 18, we have that $\det(A) = \lambda_1 \dots \lambda_n$ and $\operatorname{trace}(A) = \lambda_1 + \cdots + \lambda_n$.

Proposition 3.5

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix.

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Proof. Since A is symmetric, Corollary 2.4 of Lecture Notes 20 guarantees that it has n real eigenvalues (with algebraic multiplicities taken into account). So, let $\{\lambda_1, \dots, \lambda_n\}$ be the spectrum of A. By Theorem 2.11 of Lecture Notes 18, we have that $\det(A) = \lambda_1 \dots \lambda_n$ and $\operatorname{trace}(A) = \lambda_1 + \cdots + \lambda_n$. By Theorem 3.4, all eigenvalues of a positive semi-definite matrix are non-negative, and it follows that (a) holds.

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(a) If A is positive semi-definite, then $\det(A)$ and $\operatorname{trace}(A)$ are both non-negative.

(b) If A is positive definite, then $\det(A)$ and $\operatorname{trace}(A)$ are both positive.

Proof. Since A is symmetric, Corollary 2.4 of Lecture Notes 20 guarantees that it has n real eigenvalues (with algebraic multiplicities taken into account). So, let $\{\lambda_1, \dots, \lambda_n\}$ be the spectrum of A. By Theorem 2.11 of Lecture Notes 18, we have that $\det(A) = \lambda_1 \dots \lambda_n$ and $\operatorname{trace}(A) = \lambda_1 + \cdots + \lambda_n$. By Theorem 3.4, all eigenvalues of a positive semi-definite matrix are non-negative, and it follows that (a) holds. Similarly, by Theorem 3.3, all eigenvalues of a positive definite matrix are positive, and it follows that (b) holds. Q.E.D.

3. Methods of testing for positive definiteness

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Theorem 4.1 [Recursive test of positive definiteness]

Let n be a positive integer, and let $A = \left[\begin{array}{c|c} \alpha & \mathbf{a}^T \\ \hline \mathbf{a} & A' \end{array}\right]$ (with $\alpha \in \mathbb{R}$, $\mathbf{a} \in \mathbb{R}^n$, and $A' \in \mathbb{R}^{n \times n}$) be a symmetric matrix in $\mathbb{R}^{(n+1)\times(n+1)}$. Then A is positive-definite iff $\alpha > 0$ and $A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T$ is positive definite.

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- Theorem 4.1 allows us to reduce the problem of checking whether a symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive definite to the problem of checking whether a 1×1 matrix (obtained in $n-1$ steps, via the reduction from Theorem 4.1) is positive definite.
  - Obviously, a matrix in $\mathbb{R}^{1\times 1}$ is positive definite iff its unique entry is positive.

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- Theorem 4.1 allows us to reduce the problem of checking whether a symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive definite to the problem of checking whether a 1×1 matrix (obtained in $n-1$ steps, via the reduction from Theorem 4.1) is positive definite.
  - Obviously, a matrix in $\mathbb{R}^{1 \times 1}$ is positive definite iff its unique entry is positive.
- Before proving the theorem, let's take a look at an example.

Example 4.2

Using Theorem 4.1, determine whether the matrix

$$A \; := \; \begin{bmatrix} 4 & -2 & 4 \\ -2 & 10 & 1 \\ 4 & 1 & 6 \end{bmatrix}$$

is positive definite.

Solution.

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is positive definite.

Solution. We apply Theorem 4.1 twice. First, set $\alpha_2 := 4$, $\mathbf{a}_2 := \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, and $A_2' := \begin{bmatrix} 10 & 1 \\ 1 & 6 \end{bmatrix}$, so that $A = \left[\begin{array}{c|c} \alpha_2 & \mathbf{a}_2^T \\ \hline \mathbf{a}_2 & A_2' \end{array}\right]$.
We have that $\alpha_2 > 0$, and so by Theorem 4.2, A is positive definite iff $A_2 := A_2' - \frac{1}{\alpha_2}\mathbf{a}_2\mathbf{a}_2^T$ is positive definite.

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We have that $\alpha_2 > 0$, and so by Theorem 4.2, A is positive definite iff $A_2 := A_2' - \frac{1}{\alpha_2}\mathbf{a}_2\mathbf{a}_2^T$ is positive definite. We compute

$$\begin{aligned} A_2 \;&=\; A_2' - \tfrac{1}{\alpha_2}\mathbf{a}_2\mathbf{a}_2^T \;=\; \begin{bmatrix} 10 & 1 \\ 1 & 6 \end{bmatrix} - \tfrac{1}{4}\begin{bmatrix} -2 \\ 4 \end{bmatrix}\begin{bmatrix} -2 & 4 \end{bmatrix} \\ &=\; \begin{bmatrix} 9 & 3 \\ 3 & 2 \end{bmatrix}. \end{aligned}$$

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Using Theorem 4.1, determine whether the matrix

$$A \quad := \quad \begin{bmatrix} 4 & -2 & 4 \\ -2 & 10 & 1 \\ 4 & 1 & 6 \end{bmatrix}$$

is positive definite.

Solution (continued). Reminder: We need to determine if $A_2 = \begin{bmatrix} 9 & 3 \\ 3 & 2 \end{bmatrix}$ is positive definite.

Example 4.2

Using Theorem 4.1, determine whether the matrix

$$A := \begin{bmatrix} 4 & -2 & 4 \\ -2 & 10 & 1 \\ 4 & 1 & 6 \end{bmatrix}$$

is positive definite.

Solution (continued). Reminder: We need to determine if
$A_2 = \begin{bmatrix} 9 & 3 \\ 3 & 2 \end{bmatrix}$ is positive definite.

Set $\alpha_1 := 9$, $\mathbf{a}_1 := [\ 3\ ]$, and $A_1' := [\ 2\ ]$, so that
$A_2 = \left[\begin{array}{c|c} \alpha_1 & \mathbf{a}_1^T \\ \hline \mathbf{a}_1 & A_1' \end{array}\right]$. We have that $\alpha_1 > 0$, and so by Theorem 4.1,
A_2 is positive definite iff $A_1 := A_1' - \frac{1}{\alpha_1}\mathbf{a}_1\mathbf{a}_1^T$ is positive definite. We compute (next slide):

Example 4.2

Using Theorem 4.1, determine whether the matrix

$$A \;:=\; \begin{bmatrix} 4 & -2 & 4 \\ -2 & 10 & 1 \\ 4 & 1 & 6 \end{bmatrix}$$

is positive definite.

Solution (continued).

$$\begin{aligned} A_1 &= A_1' - \tfrac{1}{\alpha_1}\mathbf{a}_1\mathbf{a}_1^T \\ &= \begin{bmatrix} 2 \end{bmatrix} - \tfrac{1}{9}\begin{bmatrix} 3 \end{bmatrix}\begin{bmatrix} 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \end{bmatrix}. \end{aligned}$$

Since the only entry of A_1 is positive, we see that A_1 is positive definite. So, A is positive definite.

Theorem 4.1 [Recursive test of positive definiteness]

Let n be a positive integer, and let $A = \left[\begin{array}{c|c} \alpha & \mathbf{a}^T \\ \hline \mathbf{a} & A' \end{array}\right]$ (with $\alpha \in \mathbb{R}$, $\mathbf{a} \in \mathbb{R}^n$, and $A' \in \mathbb{R}^{n \times n}$) be a symmetric matrix in $\mathbb{R}^{(n+1)\times(n+1)}$. Then A is positive-definite iff $\alpha > 0$ and $A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T$ is positive definite.

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Let us first check that $A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T$ is symmetric. Since A is symmetric, so is A'. But now

$$\left(A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T\right)^T = A'^T - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T = A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T,$$

and so $A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T$ is indeed symmetric.

Proof (continued). Now, fix any $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Then

$$\begin{aligned}
\mathbf{x}^T(A' - \tfrac{1}{\alpha}\mathbf{a}\mathbf{a}^T)\mathbf{x} &= \mathbf{x}^T A'\mathbf{x} - \tfrac{1}{\alpha}(\mathbf{x}^T\mathbf{a}\mathbf{a}^T\mathbf{x}) \\
&= \left[\begin{array}{c|c} -\frac{1}{\alpha}\mathbf{a}^T\mathbf{x} & \mathbf{x}^T \end{array}\right] \left[\begin{array}{c|c} \alpha & \mathbf{a}^T \\ \hline \mathbf{a} & A' \end{array}\right] \underbrace{\left[\begin{array}{c} -\frac{1}{\alpha}\mathbf{a}^T\mathbf{x} \\ \hline \mathbf{x} \end{array}\right]}_{:=\mathbf{y}} \\
&= \mathbf{y}^T A\mathbf{y} \\
&\overset{(*)}{>} 0
\end{aligned}$$

where (*) follows from the fact that A is positive definite and $\mathbf{y} \neq \mathbf{0}$ (since $\mathbf{x} \neq \mathbf{0}$).

Proof (continued). Suppose now that $\alpha > 0$ and $A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T$ is positive definite. We must show that A is positive definite.

Proof (continued). Suppose now that $\alpha > 0$ and $A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T$ is positive definite. We must show that A is positive definite. Fix any $\mathbf{x} \in \mathbb{R}^{n+1}$, and set $\mathbf{x} = \left[\begin{array}{c|c} x_0 & \mathbf{y}^T \end{array}\right]^T$, where $x_0 \in \mathbb{R}$ and $\mathbf{y} \in \mathbb{R}^n$. We now compute:

$$\begin{aligned}
\mathbf{x}^T A\mathbf{x} &= \left[\begin{array}{c|c} x_0 & \mathbf{y}^T \end{array}\right] \left[\begin{array}{c|c} \alpha & \mathbf{a}^T \\ \hline \mathbf{a} & A' \end{array}\right] \left[\begin{array}{c} x_0 \\ \hline \mathbf{y} \end{array}\right] \\
&= \alpha x_0^2 + x_0\mathbf{a}^T\mathbf{y} + x_0\mathbf{y}^T\mathbf{a} + \mathbf{y}^T A'\mathbf{y} \\
&\overset{(*)}{=} \alpha x_0^2 + 2x_0\mathbf{a}^T\mathbf{y} + \mathbf{y}^T A'\mathbf{y} \\
&= \mathbf{y}^T(A' - \tfrac{1}{\alpha}\mathbf{a}\mathbf{a}^T)\mathbf{y} + \tfrac{1}{\alpha}\mathbf{y}^T\mathbf{a}\mathbf{a}^T\mathbf{y} + 2x_0\mathbf{a}^T\mathbf{y} + \alpha x_0^2 \\
&= \mathbf{y}^T(A' - \tfrac{1}{\alpha}\mathbf{a}\mathbf{a}^T)\mathbf{y} + (\tfrac{1}{\sqrt{\alpha}}\mathbf{a}^T\mathbf{y})^2 + 2x_0\mathbf{a}^T\mathbf{y} + (\sqrt{\alpha}x_0)^2 \\
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Proof (continued). Reminder: $\mathbf{x} = [\ x_0 \mid \mathbf{y}^T\ ]^T$; $A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T$ is positive definite; $\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T(A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T)\mathbf{y} + \left(\frac{1}{\sqrt{\alpha}}\mathbf{a}^T\mathbf{y} + \sqrt{\alpha}x_0\right)^2$.

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We now have that

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where in (*), we used the fact that $x_0\mathbf{y}^T\mathbf{a}$ is a 1×1 (and consequently symmetric) matrix, and so $x_0\mathbf{y}^T\mathbf{a} = (x_0\mathbf{y}^T\mathbf{a})^T = x_0\mathbf{a}^T\mathbf{y}$; and where for the inequality (**), we used the fact that $\mathbf{y}^T(A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T)\mathbf{y} \geq 0$, since A is positive definite.

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Proof (continued). Reminder: $\mathbf{x} = [\ x_0 \mid \mathbf{y}^T\ ]^T$; $A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T$ is positive definite; $\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T(A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T)\mathbf{y} + \left(\frac{1}{\sqrt{\alpha}}\mathbf{a}^T\mathbf{y} + \sqrt{\alpha}x_0\right)^2$.

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Proof (continued). Reminder: $\mathbf{x} = [\ x_0 \mid \mathbf{y}^T\ ]^T$; $A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T$ is positive definite; $\mathbf{x}^T A \mathbf{x} = \mathbf{y}^T(A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T)\mathbf{y} + \left(\frac{1}{\sqrt{\alpha}}\mathbf{a}^T\mathbf{y} + \sqrt{\alpha}x_0\right)^2$.

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Proof (continued). Reminder: $\mathbf{x} = [\ x_0 \mid \mathbf{y}^T\ ]^T$; $A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T$ is positive definite; $\mathbf{x}^T A \mathbf{x} = \mathbf{y}^T(A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T)\mathbf{y} + \left(\frac{1}{\sqrt{\alpha}}\mathbf{a}^T\mathbf{y} + \sqrt{\alpha}x_0\right)^2$.

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Proof (continued). Reminder: $\mathbf{x} = [\ x_0 \mid \mathbf{y}^T\ ]^T$; $A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T$ is positive definite; $\mathbf{x}^T A \mathbf{x} = \mathbf{y}^T(A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T)\mathbf{y} + \left(\frac{1}{\sqrt{\alpha}}\mathbf{a}^T\mathbf{y} + \sqrt{\alpha}x_0\right)^2$.

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where in (*), we used the fact that $x_0\mathbf{y}^T\mathbf{a}$ is a 1×1 (and consequently symmetric) matrix, and so $x_0\mathbf{y}^T\mathbf{a} = (x_0\mathbf{y}^T\mathbf{a})^T = x_0\mathbf{a}^T\mathbf{y}$; and where for the inequality (**), we used the fact that $\mathbf{y}^T(A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T)\mathbf{y} \geq 0$, since A is positive definite.

It remains to show that the inequality (**) is an equality iff $\mathbf{x} = \mathbf{0}$. If $\mathbf{x} = \mathbf{0}$, then $x_0 = 0$ and $\mathbf{y} = \mathbf{0}$, and it is obvious that the inequality (**) is an equality. Suppose now that the inequality (**) is an equality. Then $\mathbf{y}^T(A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T)\mathbf{y} = 0$ and $\frac{1}{\sqrt{\alpha}}\mathbf{a}^T\mathbf{y} + \sqrt{\alpha}x_0 = 0$. The former implies that $\mathbf{y} = \mathbf{0}$ (since $A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T$ is positive definite). But now since $\frac{1}{\sqrt{\alpha}}\mathbf{a}^T\mathbf{y} + \sqrt{\alpha}x_0 = 0$, we deduce that $x_0 = 0$.

Proof (continued). Reminder: $\mathbf{x} = [\ x_0 \mid \mathbf{y}^T\ ]^T$; $A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T$ is positive definite; $\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T(A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T)\mathbf{y} + \left(\frac{1}{\sqrt{\alpha}}\mathbf{a}^T\mathbf{y} + \sqrt{\alpha}x_0\right)^2$.

We now have that

$$\mathbf{x}^T A\mathbf{x} \;=\; \mathbf{y}^T(A' - \tfrac{1}{\alpha}\mathbf{a}\mathbf{a}^T)\mathbf{y} + \left(\tfrac{1}{\sqrt{\alpha}}\mathbf{a}^T\mathbf{y} + \sqrt{\alpha}x_0\right)^2 \;\overset{(**)}{\geq}\; 0,$$

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Proof (continued). Reminder: $\mathbf{x} = [\ x_0 \mid \mathbf{y}^T\ ]^T$; $A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T$ is positive definite; $\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T(A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T)\mathbf{y} + \left(\frac{1}{\sqrt{\alpha}}\mathbf{a}^T\mathbf{y} + \sqrt{\alpha}x_0\right)^2$.

We now have that

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Theorem 4.1 [Recursive test of positive definiteness]

Let n be a positive integer, and let $A = \left[\begin{array}{c|c} \alpha & \mathbf{a}^T \\ \hline \mathbf{a} & A' \end{array}\right]$ (with $\alpha \in \mathbb{R}$, $\mathbf{a} \in \mathbb{R}^n$, and $A' \in \mathbb{R}^{n\times n}$) be a symmetric matrix in $\mathbb{R}^{(n+1)\times(n+1)}$. Then A is positive-definite iff $\alpha > 0$ and $A' - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T$ is positive definite.

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- Theorem 4.1 allows us to reduce the problem of checking whether a symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive definite to the problem of checking whether a 1×1 matrix (obtained in $n-1$ steps, via the reduction from Theorem 4.1) is positive definite.
    - Obviously, a matrix in $\mathbb{R}^{1\times 1}$ is positive definite iff its unique entry is positive.

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Corollary 4.3 [Gaussian elimination test of positive definiteness]

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then A is positive definite iff the following sequence of $n-1$ steps can be performed and it transforms the matrix A into an upper triangular matrix with a positive main diagonal:

For $j \in \{1, \ldots, n-1\}$:

> ***Step** j**:*** *For each $i \in \{j+1, \ldots, n\}$, add a suitable scalar multiple of the j-th row to the i-th row so that the (i, j)-th entry of the matrix becomes zero.*

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- First an example, then a proof.

Example 4.4

Using Corollary 4.3, determine whether the matrix

$$A := \begin{bmatrix} 4 & -2 & 4 \\ -2 & 10 & 1 \\ 4 & 1 & 6 \end{bmatrix}$$

is positive definite.

Solution.

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$$\begin{bmatrix} 4 & -2 & 4 \\ 0 & 9 & 3 \\ 0 & 3 & 2 \end{bmatrix}.$$

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Step 2 of Corollary 4.3 can be performed: we perform the elementary row operation $R_3 \to R_3 - \frac{3}{9}R_2$, and we obtain the matrix (next slide):

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is positive definite.

Solution (continued).

$$\begin{bmatrix} 4 & -2 & 4 \\ 0 & 9 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 4.4

Using Corollary 4.3, determine whether the matrix

$$A := \begin{bmatrix} 4 & -2 & 4 \\ -2 & 10 & 1 \\ 4 & 1 & 6 \end{bmatrix}$$

is positive definite.

Solution (continued).

$$\begin{bmatrix} 4 & -2 & 4 \\ 0 & 9 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have now obtained an upper triangular matrix with a positive main diagonal. So, by Corollary 4.3, A is positive definite.

Corollary 4.3 [Gaussian elimination test of positive definiteness]

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For $j \in \{1, \ldots, n-1\}$:

> ***Step** j:* *For each $i \in \{j+1, \ldots, n\}$, add a suitable scalar multiple of the j-th row to the i-th row so that the (i, j)-th entry of the matrix becomes zero.*

Proof.

Corollary 4.3 [Gaussian elimination test of positive definiteness]

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Then A is positive definite iff the following sequence of $n-1$ steps can be performed and it transforms the matrix A into an upper triangular matrix with a positive main diagonal:

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> ***Step*** ***j:*** *For each $i \in \{j+1, \ldots, n\}$, add a suitable scalar multiple of the j-th row to the i-th row so that the (i, j)-th entry of the matrix becomes zero.*

Proof. We may assume inductively that the theorem is true for symmetric matrices in $\mathbb{R}^{n' \times n'}$, for all $n' \in \{1, \ldots, n-1\}$.

Proof (solution). Now, fix a symmetric matrix $A = [\ a_{i,j}\ ]_{n\times n}$ in $\mathbb{R}^{n\times n}$.

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Proof (solution). Now, fix a symmetric matrix $A = [\ a_{i,j}\ ]_{n \times n}$ in $\mathbb{R}^{n \times n}$. Suppose first that $a_{1,1} \leq 0$. In this case, A is not positive definite (by Proposition 3.1(b)), *Proof (solution).* Now, fix a symmetric matrix $A = [\ a_{i,j}\ ]_{n\times n}$ in $\mathbb{R}^{n\times n}$. Suppose first that $a_{1,1} \leq 0$. In this case, A is not positive definite (by Proposition 3.1(b)), and our sequence of steps either cannot be performed, or it can be performed but produces a matrix whose main diagonal has at least one negative or zero entry (this is because the $(1,1)$-th entry remains unchanged throughout, and by supposition, $a_{1,1} \leq 0$).

Proof (solution). Now, fix a symmetric matrix $A = [\ a_{i,j}\ ]_{n\times n}$ in $\mathbb{R}^{n\times n}$. Suppose first that $a_{1,1} \leq 0$. In this case, A is not positive definite (by Proposition 3.1(b)), and our sequence of steps either cannot be performed, or it can be performed but produces a matrix whose main diagonal has at least one negative or zero entry (this is because the $(1,1)$-th entry remains unchanged throughout, and by supposition, $a_{1,1} \leq 0$).

From now on, we may assume that $a_{1,1} > 0$.

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From now on, we may assume that $a_{1,1} > 0$. If $n = 1$, then $A = [\ a_{1,1}\ ]$ is positive definite, and our sequence of $n-1$ steps is empty and produces the matrix A itself, which is indeed in upper triangular with a positive main diagonal.

Proof (continued). From now on, we may assume that $n \geq 2$.

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- $R_2 \to R_2 - \frac{a_{2,1}}{a_{1,1}} R_1$;

  $\vdots$

- $R_n \to R_n - \frac{a_{n,1}}{a_{1,1}} R_1$.

(This transforms entries $2, \dots, n-1$ of the first column into 0).

Proof (continued). From now on, we may assume that $n \geq 2$. In this case, Step 1 can be performed, and it consists of the following $n-1$ elementary row operations:

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(This transforms entries $2, \dots, n-1$ of the first column into 0). But note that if we write our original matrix A in the form $A = \left[\begin{array}{c|c} a_{1,1} & \mathbf{a}^T \\ \hline \mathbf{a} & A_{1,1} \end{array}\right]$, then the matrix that we obtain after the Step 1 is precisely the matrix

$$\left[\begin{array}{c|c} a_{1,1} & \mathbf{a}^T \\ \hline \mathbf{0} & A_{1,1} - \frac{1}{a_{1,1}} \mathbf{a}\mathbf{a}^T \end{array}\right].$$

Proof (continued). From now on, we may assume that $n \geq 2$. In this case, Step 1 can be performed, and it consists of the following $n-1$ elementary row operations:

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By Theorem 4.1, A is positive definite iff $A_{1,1} - \frac{1}{a_{1,1}}\mathbf{a}\mathbf{a}^T$ is positive definite. The result now follows immediately from the induction hypothesis. Q.E.D.

- Given any $n \times n$ matrix A, and any index $k \in \{1, \ldots, n\}$, we let $A^{(k)}$ be the $k \times k$ matrix in the upper left corner of A.

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then we have that

$$A^{(1)} = \begin{bmatrix} 1 \end{bmatrix}, \qquad A^{(2)} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, \qquad A^{(3)} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

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- Clearly, for any A is an $n \times n$ matrix A, we have that $A^{(n)} = A$.

Theorem 4.5 [Sylvester's criterion of positive definiteness]

For all symmetric matrices $A \in \mathbb{R}^{n \times n}$, the following are equivalent:

(i) A is positive definite;

(ii) $\det(A^{(1)}), \ldots, \det(A^{(n)}) > 0$.

- First an example, then a proof.

Example 4.6

Using Sylvester's criterion of positive definiteness, determine whether the matrix

$$A \;:=\; \begin{bmatrix} 4 & -2 & 4 \\ -2 & 10 & 1 \\ 4 & 1 & 6 \end{bmatrix}$$

is positive definite.

Solution.

Example 4.6

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is positive definite.

Solution. First, we have that

- $A^{(1)} = \begin{bmatrix} 4 \end{bmatrix}$;
- $A^{(2)} = \begin{bmatrix} 4 & -2 \\ -2 & 10 \end{bmatrix}$;
- $A^{(3)} = A$.

We compute $\det(A^{(1)}) = 4$, $\det(A^{(2)}) = 36$, and $\det(A^{(3)}) = 36$. All three determinants are positive, and so A is positive definite.

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Proof. Suppose first that (i) holds.

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Proof. Suppose first that (i) holds. Fix an index $k \in \{1, \ldots, n\}$ and a vector $\mathbf{x}_k = [\begin{array}{ccc} x_1 & \ldots & x_k \end{array}]^T$ in $\mathbb{R}^k \setminus \{\mathbf{0}\}$.

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$$\mathbf{x}_k^T A^{(k)} \mathbf{x}_k \;=\; \mathbf{x}^T A \mathbf{x} \;\stackrel{(*)}{>}\; 0,$$

where (*) follows from the fact that A is positive definite and $\mathbf{x} \neq \mathbf{0}$ (because $\mathbf{x}_k \neq \mathbf{0}$). So, $A^{(k)}$ is positive definite. But now Proposition 3.5(b) guarantees that $\det(A^{(k)}) > 0$, and it follows that (ii) holds.

Theorem 4.5 [Sylvester's criterion of positive definiteness]

For all symmetric matrices $A \in \mathbb{R}^{n\times n}$, the following are equivalent:

1. (i) A is positive definite;
2. (ii) $\det(A^{(1)}), \ldots, \det(A^{(n)}) > 0$.

Proof (continued). Suppose now that (ii) holds.

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(i) A is positive definite;

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Proof (continued). Suppose now that (ii) holds. Since $\det(A^{(1)}) > 0$, we know that $a_{1,1} > 0$.

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Proof (continued). Suppose now that (ii) holds. Since $\det(A^{(1)}) > 0$, we know that $a_{1,1} > 0$. If $n = 1$, it follows that $A = [\ a_{1,1}\ ]$ is positive definite (because its only entry is positive).

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Proof (continued). Suppose now that (ii) holds. Since $\det(A^{(1)}) > 0$, we know that $a_{1,1} > 0$. If $n = 1$, it follows that $A = \left[\ a_{1,1}\ \right]$ is positive definite (because its only entry is positive). So, we may assume that $n \geq 2$. We now start performing the steps described in Corollary 4.3, and we proceed until we either complete all $n - 1$ steps, or until a step cannot be performed.

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Proof (continued). Suppose now that (ii) holds. Since $\det(A^{(1)}) > 0$, we know that $a_{1,1} > 0$. If $n = 1$, it follows that $A = \begin{bmatrix} a_{1,1} \end{bmatrix}$ is positive definite (because its only entry is positive). So, we may assume that $n \geq 2$. We now start performing the steps described in Corollary 4.3, and we proceed until we either complete all $n-1$ steps, or until a step cannot be performed. Since $n \geq 2$ and $a_{1,1} \neq 0$, Step 1 can be performed.

Theorem 4.5 [Sylvester's criterion of positive definiteness]

For all symmetric matrices $A \in \mathbb{R}^{n \times n}$, the following are equivalent:

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(ii) $\det(A^{(1)}), \dots, \det(A^{(n)}) > 0$.

Proof (continued). Suppose now that (ii) holds. Since $\det(A^{(1)}) > 0$, we know that $a_{1,1} > 0$. If $n = 1$, it follows that $A = \left[\ a_{1,1}\ \right]$ is positive definite (because its only entry is positive). So, we may assume that $n \geq 2$. We now start performing the steps described in Corollary 4.3, and we proceed until we either complete all $n-1$ steps, or until a step cannot be performed. Since $n \geq 2$ and $a_{1,1} \neq 0$, Step 1 can be performed. Now, let Step ℓ be the last step that we perform, and let $B = \left[\ b_{i,j}\ \right]_{n \times n}$ be the matrix that we obtain after performing our ℓ steps.

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