Linear Algebra 2

Lecture #20
The Jordan normal form. Symmetric matrices and orthogonal diagonalization

Irena Penev

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    1. the Jordan normal form;

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    2. symmetric matrices and orthogonal diagonalization.

1. The Jordan normal form

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Definition

Given a field $\mathbb{F}$, a scalar $\lambda_0 \in \mathbb{F}$, and a positive integer k, the *Jordan block* $J_k(\lambda_0)$ is defined to be the matrix

$$J_k(\lambda_0) = \begin{bmatrix} \lambda_0 & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_0 & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_0 \end{bmatrix}_{k \times k} .$$

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- Thus, $J_k(\lambda_0)$ is a matrix in $\mathbb{F}^{k\times k}$, it has all λ_0's on the main diagonal, all 1's on the diagonal right above the main diagonal, and 0's everywhere else.

- For example:
  - $J_1(\lambda_0) = \begin{bmatrix} \lambda_0 \end{bmatrix}$;
  - $J_2(\lambda_0) = \begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix}$;
  - $J_3(\lambda_0) = \begin{bmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 1 \\ 0 & 0 & \lambda_0 \end{bmatrix}$;
  - $J_4(\lambda_0) = \begin{bmatrix} \lambda_0 & 1 & 0 & 0 \\ 0 & \lambda_0 & 1 & 0 \\ 0 & 0 & \lambda_0 & 1 \\ 0 & 0 & 0 & \lambda_0 \end{bmatrix}$;
  - $J_5(\lambda_0) = \begin{bmatrix} \lambda_0 & 1 & 0 & 0 & 0 \\ 0 & \lambda_0 & 1 & 0 & 0 \\ 0 & 0 & \lambda_0 & 1 & 0 \\ 0 & 0 & 0 & \lambda_0 & 1 \\ 0 & 0 & 0 & 0 & \lambda_0 \end{bmatrix}$.

Definition

A *Jordan matrix* (also called a matrix in *Jordan normal form*) is a square matrix that has Jordan blocks along the main diagonal, and has 0's everywhere else. Thus, a Jordan matrix is a matrix of the form

$$\begin{bmatrix} J_{k_1}(\lambda_1) & O & \dots & O \\ O & J_{k_2}(\lambda_2) & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & J_{k_\ell}(\lambda_\ell) \end{bmatrix},$$

where $\lambda_1, \dots, \lambda_\ell$ are scalars in $\mathbb{F}$, $k_1, \dots, k_\ell$ are positive integers, and the O's are zero matrices of appropriate sizes.

- For instance, the following is a Jordan matrix with four Jordan blocks, namely $J_3(3)$, $J_2(2)$, $J_1(2)$, and $J_3(3)$:

$$\begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

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  - This is because (by Theorem 4.3 of Lecture Notes 10) similar matrices represent the same linear transformation, only with respect to (possibly) different bases.
  - A change in the order of Jordan blocks corresponds to a change in the order of basis elements.

- For instance, suppose that V is a finite-dimensional vector space over a field $\mathbb{F}$, that $f : V \to V$ is a linear transformation, and that $\mathcal{B} = \{\mathbf{a}_1, \dots, \mathbf{a}_{k_1}, \mathbf{b}_1, \dots, \mathbf{b}_{k_2}, \mathbf{c}_1, \dots, \mathbf{c}_{k_3}, \mathbf{d}_1, \dots, \mathbf{d}_{k_4}\}$ (with $k_1, k_2, k_3, k_4 \geq 1$) is a basis of V such that

$$
{}_{\mathcal{B}}[f]_{\mathcal{B}} \quad = \quad \begin{bmatrix} J_{k_1}(\lambda_1) & O & O & O \\ O & J_{k_2}(\lambda_2) & O & O \\ O & O & J_{k_3}(\lambda_3) & O \\ O & O & O & J_{k_4}(\lambda_4) \end{bmatrix}.
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- Then for the basis $\mathcal{C} = \{\mathbf{b}_1, \ldots, \mathbf{b}_{k_2}, \mathbf{d}_1, \ldots, \mathbf{d}_{k_4}, \mathbf{a}_1, \ldots, \mathbf{a}_{k_1}, \mathbf{c}_1, \ldots, \mathbf{c}_{k_3}\}$, we have

$$ {}_{\mathcal{C}}[f]_{\mathcal{C}} = \begin{bmatrix} J_{k_2}(\lambda_2) & O & O & O \\ O & J_{k_4}(\lambda_4) & O & O \\ O & O & J_{k_1}(\lambda_1) & O \\ O & O & O & J_{k_3}(\lambda_3) \end{bmatrix}. $$

- For instance, suppose that V is a finite-dimensional vector space over a field $\mathbb{F}$, that $f : V \to V$ is a linear transformation, and that $\mathcal{B} = \{\mathbf{a}_1, \dots, \mathbf{a}_{k_1}, \mathbf{b}_1, \dots, \mathbf{b}_{k_2}, \mathbf{c}_1, \dots, \mathbf{c}_{k_3}, \mathbf{d}_1, \dots, \mathbf{d}_{k_4}\}$ (with $k_1, k_2, k_3, k_4 \geq 1$) is a basis of V such that

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- Then for the basis $\mathcal{C} = \{\mathbf{b}_1, \dots, \mathbf{b}_{k_2}, \mathbf{d}_1, \dots, \mathbf{d}_{k_4}, \mathbf{a}_1, \dots, \mathbf{a}_{k_1}, \mathbf{c}_1, \dots, \mathbf{c}_{k_3}\}$, we have

$$_{\mathcal{C}}[f]_{\mathcal{C}} = \begin{bmatrix} J_{k_2}(\lambda_2) & O & O & O \\ O & J_{k_4}(\lambda_4) & O & O \\ O & O & J_{k_1}(\lambda_1) & O \\ O & O & O & J_{k_3}(\lambda_3) \end{bmatrix}.$$

- Clearly, the two Jordan matrices above are similar, since $_{\mathcal{B}}[f]_{\mathcal{B}}$ and $_{\mathcal{C}}[f]_{\mathcal{C}}$ are similar.

Theorem 1.1

Assume that $\mathbb{F}$ is an **algebraically closed field**, and let $A \in \mathbb{F}^{n\times n}$ be a matrix. Then there exist scalars $\lambda_1, \ldots, \lambda_\ell \in \mathbb{F}$ and positive integers $k_1, \ldots, k_\ell$ such that $k_1 + \cdots + k_\ell = n$ and such that A is similar to the matrix

$$\begin{bmatrix} J_{k_1}(\lambda_1) & O & \ldots & O \\ O & J_{k_2}(\lambda_2) & \ldots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \ldots & J_{k_\ell}(\lambda_\ell) \end{bmatrix},$$

called the *Jordan normal form* of A. The scalars $\lambda_1, \ldots, \lambda_\ell$ and integers $k_1, \ldots, k_\ell$ are unique up to a reordering of the λ_i's and the corresponding k_i's.

Proof. Omitted.

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  1. Theorem 1.1 only holds for **algebraically closed fields**.
     - The only algebraically closed field that we have seen is $\mathbb{C}$, but others exist.
  2. Two matrices are similar if and only if they have the same Jordan normal form, up to a reordering of the Jordan blocks.
     - Indeed, as we saw above, any two Jordan matrices with the same Jordan blocks (counting repetitions) are similar.
     - The other direction follows from the uniqueness part of Theorem 1.1.

- **Remarks:**
  3. Since every Jordan matrix is upper triangular, its eigenvalues, together with their algebraic multiplicities, can easily be read off the Jordan matrix itself: the eigenvalues are precisely the entries along the main diagonal of the Jordan matrix, and the algebraic multiplicity of each eigenvalue is the number of times that it appears on the main diagonal. For instance, the eigenvalues of the Jordan matrix

$$\begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

  are 3 (with algebraic multiplicity 6) and 2 (with algebraic multiplicity 3).

- **Remarks:**
  4. The geometric multiplicity of each eigenvalue of a Jordan matrix J can also easily be read off the Jordan matrix J: the geometric multiplicity of each eigenvalue λ is precisely the number of Jordan blocks of the form $J_k(\lambda)$ that appear along the main diagonal of J. For instance, for the Jordan matrix from the previous slide, the geometric multiplicity of the eigenvalue 3 is 2, and the geometric multiplicity of the eigenvalue 2 is also 2.

- **Remarks:**
  4. The geometric multiplicity of each eigenvalue of a Jordan matrix J can also easily be read off the Jordan matrix J: the geometric multiplicity of each eigenvalue λ is precisely the number of Jordan blocks of the form $J_k(\lambda)$ that appear along the main diagonal of J. For instance, for the Jordan matrix from the previous slide, the geometric multiplicity of the eigenvalue 3 is 2, and the geometric multiplicity of the eigenvalue 2 is also 2.
     - Recall that similar matrices have the same eigenvalues, with the same corresponding algebraic (respectively, geometric) multiplicities. So, if we know the Jordan normal form of a matrix A, then we can easily read off the eigenvalues of A, together with their algebraic and geometric multiplicities.

- **Remarks:**
  5. For the case when $\mathbb{F}$ is an algebraically closed field, Theorem 1.1 yields another proof of the fact that the geometric multiplicity of an eigenvalue λ of a matrix $A \in \mathbb{F}^{n\times n}$ is no greater than the algebraic multiplicity of λ: for the Jordan normal form J of A, the number of Jordan blocks of the form $J_k(\lambda)$ that appear along the main diagonal of J is no greater than the total number of times that λ appears on the main diagonal of the Jordan matrix J.

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     - If $\mathbb{F}$ is not algebraically closed, this can be handled by first extending $\mathbb{F}$ to an algebraically closed field (for example, extending $\mathbb{R}$ to $\mathbb{C}$), and then considering the Jordan normal form of A, now considered as a matrix whose entries come from this algebraically closed field.

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        - Any field $\mathbb{F}$ can be extended in this way, but the details are beyond the scope of this course.

Example 1.2

Let $A_1, A_2, A_3 \in \mathbb{C}^{7\times 7}$ be matrices whose Jordan normal forms are, respectively, the matrices J_1, J_2, J_3 below.

- $J_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$
- $J_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix};$
- $J_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$

Determine which (if any) of A_1, A_2, A_3 are similar. Then, for each $i \in \{1, 2, 3\}$, find all the eigenvalues of A_i, along with their algebraic and geometric multiplicities.

Solution. We use colors to indicate the Jordan blocks of the three Jordan matrices.

- $J_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$
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We now list the Jordan blocks of J_1, J_2, J_3 (including any repetitions:

- the Jordan blocks of J_1 are $J_1(0), J_3(1), J_1(1), J_2(0)$;
- the Jordan blocks of J_2 are $J_1(1), J_2(0), J_1(0), J_3(1)$;
- the Jordan blocks of J_3 are $J_1(0), J_2(1), J_2(1), J_2(0)$.

Solution (continued). Reminder:

- the Jordan blocks of J_1 are $J_1(0), J_3(1), J_1(1), J_2(0)$;
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J_1 and J_2 have the same Jordan blocks (counting repetitions), and so A_1 and A_2 are similar.

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J_1 and J_2 have the same Jordan blocks (counting repetitions), and so A_1 and A_2 are similar. On the other hand, the Jordan blocks of the matrix J_3 are different from those of J_1 and J_2, and so A_3 is not similar to A_1 and A_2.

Solution (continued). Reminder:

- the Jordan blocks of J_1 are $J_1(0), J_3(1), J_1(1), J_2(0)$;
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Further, we see from the matrices J_1, J_2, J_3, that A_1, A_2, A_3 all have exactly two eigenvalues: the eigenvalue 0 with algebraic multiplicity 3 and geometric multiplicity 2, and the eigenvalue 1 with algebraic multiplicity 4 and geometric multiplicity 2.

Proposition 2.1 of Lecture Notes 19

Let A and B be similar matrices in $\mathbb{F}^{n\times n}$. Then A and B have the same characteristic polynomial, as well as the same eigenvalues, with the same corresponding algebraic multiplicities, and with the same corresponding geometric multiplicities.

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- The solution of Example 1.2 shows that the converse of Proposition 2.1 of Lecture Notes 19: square matrices that have the same eigenvalues, with the same corresponding algebraic and geometric multiplicities, need not be similar.

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- However, in some special cases, this can be done relatively easily, as the following example shows.

Example 1.3

Consider the matrix

$$A = \begin{bmatrix} 5 & -2 & 2 & -2 & 0 \\ 0 & 6 & -1 & 3 & 2 \\ 2 & 2 & 7 & -2 & -2 \\ 2 & 3 & 1 & 2 & -4 \\ -2 & -2 & -2 & 6 & 11 \end{bmatrix}$$

in $\mathbb{C}^{5\times 5}$. Compute the Jordan normal form of A.

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in $\mathbb{C}^{5\times 5}$. Compute the Jordan normal form of A.

Solution. First, we compute the characteristic polynomial of A: $p_A(\lambda) = \det(\lambda I_5 - A) = (\lambda - 7)^3(\lambda - 5)^2$. So, A has two eigenvalues: $\lambda_1 = 7$ (with algebraic multiplicity 3) and $\lambda_2 = 5$ (with algebraic multiplicity 2).

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We now compute the Jordan blocks associated with each eigenvalue.

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Solution (continued). Reminder: $p_A(\lambda) = (\lambda - 7)^3(\lambda - 5)^2$.

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in $\mathbb{C}^{5\times 5}$. Compute the Jordan normal form of A.

Solution (continued). Reminder: $p_A(\lambda) = (\lambda - 7)^3(\lambda - 5)^2$.

Since the eigenvalue $\lambda_1 = 7$ has algebraic multiplicity 3, there are three possibilities: one $J_3(7)$ block; one $J_2(7)$ block and one $J_1(7)$ block; or three $J_1(7)$ blocks.

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in $\mathbb{C}^{5\times 5}$. Compute the Jordan normal form of A.

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Since the eigenvalue $\lambda_1 = 7$ has algebraic multiplicity 3, there are three possibilities: one $J_3(7)$ block; one $J_2(7)$ block and one $J_1(7)$ block; or three $J_1(7)$ blocks. But by row reducing the matrix $7I_5 - A$, we see that $\lambda_1 = 7$ is an eigenvalue of geometric multiplicity 2.

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in $\mathbb{C}^{5\times 5}$. Compute the Jordan normal form of A.

Solution (continued). Reminder: $p_A(\lambda) = (\lambda - 7)^3(\lambda - 5)^2$.

Example 1.3

Consider the matrix

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For the eigenvalue $\lambda_2 = 5$, there are two possibilities: one $J_2(5)$ block, or two $J_1(5)$ blocks.

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For the eigenvalue $\lambda_2 = 5$, there are two possibilities: one $J_2(5)$ block, or two $J_1(5)$ blocks. But by row reducing the matrix $5I_5 - A$, we see that $\lambda_1 = 5$ is an eigenvalue of geometric multiplicity 2.

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For the eigenvalue $\lambda_2 = 5$, there are two possibilities: one $J_2(5)$ block, or two $J_1(5)$ blocks. But by row reducing the matrix $5I_5 - A$, we see that $\lambda_1 = 5$ is an eigenvalue of geometric multiplicity 2. Consequently, the number of Jordan blocks associated with the eigenvalue $\lambda_2 = 5$ is two. So, we get two $J_1(5)$ blocks.

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Solution (continued). Thus, the Jordan normal form of A has four Jordan blocks, namely, $J_2(7)$, $J_1(7)$, $J_1(2)$, $J_1(2)$. The actual Jordan normal form of A can be obtained by arranging these blocks along the main diagonal in any order. For instance, either of the following matrices is the Jordan normal form of A:

- $$\begin{bmatrix} 7 & 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix};$$
- $$\begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}.$$

Theorem 1.1

Assume that $\mathbb{F}$ is an **algebraically closed field**, and let $A \in \mathbb{F}^{n\times n}$ be a matrix. Then there exist scalars $\lambda_1, \ldots, \lambda_\ell \in \mathbb{F}$ and positive integers $k_1, \ldots, k_\ell$ such that $k_1 + \cdots + k_\ell = n$ and such that A is similar to the matrix

$$\begin{bmatrix} J_{k_1}(\lambda_1) & O & \ldots & O \\ O & J_{k_2}(\lambda_2) & \ldots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \ldots & J_{k_\ell}(\lambda_\ell) \end{bmatrix},$$

called the *Jordan normal form* of A. The scalars $\lambda_1, \ldots, \lambda_\ell$ and integers $k_1, \ldots, k_\ell$ are unique up to a reordering of the λ_i's and the corresponding k_i's.

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- This reduces the problem to showing that each square matrix with exactly one eigenvalue is similar to some Jordan matrix.
- So, suppose that A has exactly one eigenvalue, say λ_0.
- We may in fact assume that $\lambda_0 = 0$, for otherwise, we consider $A' := A - \lambda_0 I_n$ instead of A.
  - If we can show that A' is similar to a Jordan matrix J, then A is similar to the Jordan matrix $J + \lambda_0 I_n$.

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- Now, let $f_A : \mathbb{F}^n \to \mathbb{F}^n$ be given by $\mathbf{x} \mapsto A\mathbf{x}$, so that A is the standard matrix of f_A.

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- Now, let $f_A : \mathbb{F}^n \to \mathbb{F}^n$ be given by $\mathbf{x} \mapsto A\mathbf{x}$, so that A is the standard matrix of f_A.
- The goal is now to construct a basis $\mathcal{B}$ such that the matrix ${}_{\mathcal{B}}[f_A]_{\mathcal{B}}$ is in Jordan normal form.

- First, we consider any basis of $\mathsf{Nul}(A^{p-1})$, and then we extend it to a basis of $\mathsf{Nul}(A^p) = \mathbb{F}^n$ using some vectors $\mathbf{u}_1, \dots, \mathbf{u}_r \in \mathsf{Nul}(A^p) \setminus \mathsf{Nul}(A^{p-1})$.

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- The first vectors of our basis $\mathcal{B}$ will be the vectors $A^{p-1}\mathbf{u}_1, A^{p-2}\mathbf{u}_1, \dots, A\mathbf{u}_1, \mathbf{u}_1, \dots, A^{p-1}\mathbf{u}_k, A^{p-2}\mathbf{u}_k, \dots, A\mathbf{u}_k, \mathbf{u}_k$.

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- Each such chain $A^{q-1}\mathbf{u}, A^{q-2}\mathbf{u}, \ldots, A\mathbf{u}, \mathbf{u}$ will produce the Jordan block $J_q(0)$ in the matrix ${}_{\mathcal{B}}[f_A]_{\mathcal{B}}$.

2. Symmetric matrices and orthogonal diagonalization

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- Recall that the *standard scalar product* on $\mathbb{C}^{n\times n}$, denoted by $\cdot$, is defined as follows: for all $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$ in $\mathbb{C}^n$, we set

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- We shall denote by $|| \cdot ||$ the norm on $\mathbb{C}^{n \times n}$ induced by the standard scalar product on $\mathbb{C}^{n \times n}$.

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- For a matrix $A = [a_{i,j}]_{n \times m}$ in $\mathbb{C}^{n \times m}$, we set $\overline{A} = [\overline{a_{i,j}}]_{n \times m}$, i.e. for the i, j-th entry of $\overline{A}$ is the $\overline{a_{i,j}}$ (the complex conjugate of $a_{i,j}$).

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- The *Hermitian transpose* of a matrix $A \in \mathbb{C}^{n\times m}$ is the matrix $A^* = (\overline{A})^T$.
- For example, if $A = \begin{bmatrix} -1+i & 3 & 2i \\ 1+2i & 4-2i & 3 \end{bmatrix}$, then
  - $\overline{A} = \begin{bmatrix} -1-i & 3 & -2i \\ 1-2i & 4+2i & 3 \end{bmatrix}$
  - $A^* = \begin{bmatrix} -1-i & 1-2i \\ 3 & 4+2i \\ -2i & 3 \end{bmatrix}$

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  is Hermitian.
- Note that all entries on the main diagonal of a Hermitian matrix are real.
- Note also that if all entries of a matrix in $\mathbb{C}^{n \times n}$ happen to be real, then that matrix is Hermitian if and only if it is symmetric.

Proposition 2.1

For all $\mathbf{x} \in \mathbb{C}^n$, we have that $\mathbf{x}^*\mathbf{x} = ||\mathbf{x}||^2$.

Proof.

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Proof. Fix a vector $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ in $\mathbb{C}^n$. Then

$$\begin{aligned}
\mathbf{x}^*\mathbf{x} &= \begin{bmatrix} \overline{x_1} & \dots & \overline{x_n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= \sum_{k=1}^{n} \overline{x_k} x_k \\
&= \sum_{k=1}^{n} x_k \overline{x_k} \\
&= \mathbf{x} \cdot \mathbf{x} \\
&= ||\mathbf{x}||^2.
\end{aligned}$$

Q.E.D.

Proposition 2.2

For all $A, B \in \mathbb{C}^{n\times m}$ and $\alpha \in \mathbb{C}$, the following hold:

1. (a) $(A^*)^* = A$;
2. (b) $(\alpha A)^* = \overline{\alpha} A^*$;
3. (c) $(A+B)^* = A^* + B^*$;
4. (d) $(AB)^* = B^* A^*$.

Proof. Exercise.

Theorem 2.3

All eigenvalues of a Hermetrian matrix are real.

- **Remark:**
  - Recall that, since the field $\mathbb{C}$ is algebraically closed, every matrix in $\mathbb{C}^{n \times n}$ has n complex eigenvalues (with algebraic multiplicities taken into account).
  - So, Theorem 2.3 states that if A is a Hermitian matrix in $\mathbb{C}^{n \times n}$, then all n eigenvalues of A (with algebraic multiplicities taken into account) are real.

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Proof. Let $A \in \mathbb{C}^{n \times n}$ be Hermitian matrix, let λ be any eigenvalue of A, and let $\mathbf{x}$ be an associated eigenvector of A. After possibly rescaling (i.e. replacing $\mathbf{x}$ by $\frac{\mathbf{x}}{||\mathbf{x}||}$), we may assume that $\mathbf{x}$ is a unit vector, i.e. that it satisfies $||\mathbf{x}|| = 1$.

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$$\begin{aligned}
\mathbf{x}^* A\mathbf{x} &= \mathbf{x}^*(\lambda\mathbf{x}) && \text{because } A\mathbf{x} = \lambda\mathbf{x} \\
&= \lambda(\mathbf{x}^*\mathbf{x}) \\
&= \lambda||\mathbf{x}||^2 && \text{by Proposition 2.1} \\
&= \lambda && \text{because } ||\mathbf{x}|| = 1.
\end{aligned}$$

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Proof (continued). Reminder: $\mathbf{x}^* A\mathbf{x} = \lambda$.

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But we now have the following:

$$\begin{aligned}
\lambda &= \mathbf{x}^* A\mathbf{x} \\
&= \mathbf{x}^* A^*\mathbf{x} && \text{because } A \text{ is Hermitian} \\
&= \mathbf{x}^* A^*(\mathbf{x}^*)^* && \text{by Proposition 2.2(a)} \\
&= (\mathbf{x}^* A\mathbf{x})^* && \text{by Proposition 2.2(d)} \\
&= \lambda^* && \begin{array}{l}\text{where we consider } \lambda \text{ as} \\ \text{a } 1 \times 1 \text{ complex matrix}\end{array} \\
&= \overline{\lambda} && \begin{array}{l}\text{where we consider } \lambda \text{ as} \\ \text{a complex number.}\end{array}
\end{aligned}$$

So, λ is a real number. Q.E.D.

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Corollary 2.4

Every symmetric matrix in $\mathbb{R}^{n\times n}$ has n real eigenvalues (with algebraic multiplicities taken into account). In other words, for every symmetric matrix $A \in \mathbb{R}^{n\times n}$, the sum of algebraic multiplicities of its distinct (real) eigenvalues is n.

Proof.

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Proof. Consider any symmetric matrix $A \in \mathbb{R}^{n\times n}$. If we consider A as a matrix in $\mathbb{C}^{n\times n}$, then A is in fact Hermitian, and so by Theorem 2.3, all n eigenvalues (with algebraic multiplicities taken into account) of A are real. Q.E.D.

- Recall that a matrix $Q \in \mathbb{R}^{n\times n}$ is *orthogonal* if $Q^T Q = I_n$.

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Theorem 2.1 of Lecture Notes 14

Let $Q \in \mathbb{R}^{n \times n}$. Then the following are equivalent:

(a) Q is orthogonal (i.e. satisfies $Q^T Q = I_n$);
(b) Q is invertible and satisfies $Q^{-1} = Q^T$;
(c) $QQ^T = I_n$;
(d) Q^T is orthogonal;
(e) Q is invertible and Q^{-1} is orthogonal;
(f) the columns of Q form an orthonormal basis of $\mathbb{R}^n$;
(g) the columns of Q^T form an orthonormal basis of $\mathbb{R}^n$.

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5. Q is invertible and Q^{-1} is orthogonal;
6. the columns of Q form an orthonormal basis of $\mathbb{R}^n$;
7. the columns of Q^T form an orthonormal basis of $\mathbb{R}^n$.

Definition

A matrix $A \in \mathbb{R}^{n \times n}$ is *orthogonally diagonalizable* if there exists a diagonal matrix D and an orthogonal matrix Q, both in $\mathbb{R}^{n \times n}$, such that $D = Q^T A Q$ (equivalently: $A = QDQ^T$).

Theorem 2.5

A matrix in $\mathbb{R}^{n \times n}$ is symmetric iff it is orthogonally diagonalizable.

Proof.

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Proof. Let us first show that orthogonally diagonalizable matrices are symmeteric.

Theorem 2.5

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Proof. Let us first show that orthogonally diagonalizable matrices are symmeteric. Fix any orthogonally diagonalizable matrix $A \in \mathbb{R}^{n\times n}$. Let D be a diagonal and Q an orthogonal matrix, both in $\mathbb{R}^{n\times n}$, such that $D = Q^T AQ$.

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$$A^T \;=\; (QDQ^T)^T \;=\; (Q^T)^T D^T Q^T \;\overset{(*)}{=}\; QDQ^T \;=\; A,$$

where in (*), we used the fact that $D^T = D$, since D is diagonal. Thus, A is symmetric.

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For $n = 1$, the result is immediate: indeed, if $A \in \mathbb{R}^{1\times 1}$, then A is diagonal, and we can take $D := A$ and $Q := I_1$.

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Now, fix a positive integer n, and assume inductively that the claim holds for symmetric matrices in $\mathbb{R}^{n\times n}$. Fix any symmetric matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$. By Corollary 2.4, A has $n+1$ real eigenvalues (with algebraic multiplicities taken into account). Let $\lambda_0 \in \mathbb{R}$ be an eigenvalue of A, and let $\mathbf{x}_0 \in \mathbb{R}^n$ be an associated eigenvector of A. After possibly rescaling (i.e. replacing $\mathbf{x}_0$ by $\frac{\mathbf{x}_0}{||\mathbf{x}_0||}$), we may assume that $||\mathbf{x}_0|| = 1$. Now, using Corollary 2.5(d) of Lecture Notes 12, we let $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ be an orthonormal basis of $\mathbb{R}^{n+1}$. Set $S := \left[\begin{array}{cccc} \mathbf{x}_0 & \mathbf{x}_1 & \dots & \mathbf{x}_n \end{array}\right]$; then S is an orthogonal matrix.

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Proof (continued). Reminder: λ_0 is an eigenvalue of the symmetric matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$ with an associated eigenvector $\mathbf{x}_0$;
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Since $\mathbf{x}_0$ is an eigenvector of A associated with the eigenvalue λ_0, we know $A\mathbf{x}_0 = \lambda_0\mathbf{x}_0$, and consequently, $(\lambda_0 I_{n+1} - A)\mathbf{x}_0 = \mathbf{0}$.

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Now, set

- $\widetilde{D} := \left[\begin{array}{c:c} 0 & \mathbf{0}^T \\ \hdashline \mathbf{0} & D_0 \end{array}\right]$;
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Since D_0 is diagonal, so are $\widetilde{D}$ and D. Since Q_0 is orthogonal, so is R.

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First of all, we have that

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S^T(\lambda_0 I_{n+1} - A)S &= \left[\begin{array}{c:c} 0 & \mathbf{0}^T \\ \hdashline \mathbf{0} & A_0 \end{array}\right] \\
&= \left[\begin{array}{c:c} 0 & \mathbf{0}^T \\ \hdashline \mathbf{0} & Q_0 D_0 Q_0^T \end{array}\right] \\
&= \left[\begin{array}{c:c} 1 & \mathbf{0}^T \\ \hdashline \mathbf{0} & Q_0 \end{array}\right] \left[\begin{array}{c:c} 0 & \mathbf{0}^T \\ \hdashline \mathbf{0} & D_0 \end{array}\right] \left[\begin{array}{c:c} 1 & \mathbf{0}^T \\ \hdashline \mathbf{0} & Q_0 \end{array}\right]^T \\
&= R\widetilde{D}R^T,
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and consequently (next slide),

Proof (continued).

$$\begin{aligned}
\widetilde{D} &= R^T S^T (\lambda_0 I_{n+1} - A) SR \\
&= (SR)^T (\lambda_0 I_{n+1} - A) SR \\
&= Q^T (\lambda_0 I_{n+1} - A) Q \\
&= Q^T (\lambda_0 I_{n+1}) Q - Q^T A Q \\
&= \lambda_0 Q^T Q - Q^T A Q \\
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It follows that

$$Q^T A Q = \lambda_0 I_{n+1} - \widetilde{D} = D,$$

and we are done. Q.E.D.

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Corollary 2.6

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix, and suppose that λ_1 and λ_2 are distinct eigenvalues of A. Then $E_{\lambda_1} \perp E_{\lambda_2}$.

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Proof (outline).

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Proof (outline). Using Theorem 2.5, we fix a diagonal matrix D and an orthogonal matrix Q, both in $\mathbb{R}^{n \times n}$, such that $D = Q^T AQ$.

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- Since all the $\mathcal{C}_\lambda$'s are orthonormal sets, and since they are moreover orthogonal to each other (by Corollary 2.6), we see that the columns of Q form an orthonormal set; since Q is an $n \times n$ matrix, its columns in fact form an orthonormal basis of $\mathbb{R}^n$, and it follows that Q is orthogonal.

Example 2.7

Orthogonally diagonalize the following symmetric matrix in $\mathbb{R}^{3\times 3}$:

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

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$\mathcal{C}_2 = \{\begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{bmatrix}^T, \begin{bmatrix} \frac{4}{3\sqrt{5}} & \frac{2}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}^T\}$ of E_{λ_2}.

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Solution (continued). We now set

- $D := \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix};$
- $Q := \begin{bmatrix} -2/3 & -1/\sqrt{5} & 4/(3\sqrt{5}) \\ -1/3 & 2/\sqrt{5} & 2/(3\sqrt{5}) \\ 2/3 & 0 & 5/(3\sqrt{5}) \end{bmatrix}.$

Now D is diagonal, Q is orthogonal, and $D = Q^T AQ$. Q.E.D.

The spectral theorem for symmetric matrices

Every symmetric matrix $A \in \mathbb{R}^{n\times n}$ satisfies all the following:

(a) A is orthogonally diagonalizable;
(b) the eigenspaces of A are pairwise orthogonal;
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Proof. (a) follows from Theorem 2.5, and (b) follows from Corollary 2.6. It remains to prove (c). By (a), there exists a diagonal matrix D and an orthogonal matrix Q, both in $\mathbb{R}^{n \times n}$, such that $D = Q^T A Q$. The columns of Q are all eigenvectors of A (this follows from the proof of Theorem 4.2 of Lecture Notes 19), and since Q is orthogonal, its columns form an orthonormal basis of $\mathbb{R}^n$. Thus, (c) holds. Q.E.D.