Linear Algebra 2

Lecture #19
The Cayley-Hamilton theorem. Diagonalization

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  1. the Cayley-Hamilton theorem;
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- In what follows, $\mathbb{F}$ is a fixed field.

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- The Cayley-Hamilton theorem essentially states that every square matrix is a root of its own characteristic polynomial.
  - Here, we need to treat the free coefficient of the characteristic polynomial as that coefficient times the identity matrix of the appropriate size.

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- For example, for a matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, we have

$$p_A(\lambda) = \det(\lambda I_2 - A) = \begin{vmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{vmatrix} = \lambda^2 - 5\lambda - 2,$$

and

$$\begin{aligned} A^2 - 5A - 2I_2 &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 - 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

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- The *cofactor matrix* of A is the matrix $[\ C_{i,j}\ ]_{n\times n}$.
- The *adjugate matrix* (also called the *classical adjoint*) of A, denoted by adj(A), is the transponse of the cofactor matrix of A, i.e.

$$\text{adj}(A) \ := \ [\ C_{i,j}\ ]_{n\times n}{}^T.$$

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Proof. Let us first show that the first statement implies the second. Indeed, if A is invertible, then $\det(A) \neq 0$, and so if the first statement holds, then we get that

$$\left(\frac{1}{\det(A)}\text{adj}(A)\right) A \;=\; I_n \;=\; A \left(\frac{1}{\det(A)}\text{adj}(A)\right),$$

and consequently, $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$.

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Fix indices $i, j \in \{1, \ldots, n\}$. The i,j-th entry of the matrix $\det(A) I_n$ is $\det(A)$ if $i = j$, and is zero if $i \neq j$. We must show this holds for the i,j-th entry of the matrices $\operatorname{adj}(A)\, A$ and $A \operatorname{adj}(A)$ as well.

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Proof. If $n = 1$, then the result is immediate. Indeed, suppose that $n = 1$, and consider any matrix $A = [a_{1,1}]$ in $\mathbb{F}^{1 \times 1}$. Then $p_A(\lambda) = \det(\lambda I_1 - A) = \det(\left[\begin{array}{c} \lambda - a_{1,1} \end{array}\right]) = \lambda - a_{1,1}$, and we see that $A - a_{1,1} I_1 = O_{1 \times 1}$.

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So, assume that $n \geq 2$. By Theorem 1.1 applied to the matrix $\lambda I_n - A$ (where λ is a variable), we get that

$$(\lambda I_n - A)\,\mathrm{adj}(\lambda I_n - A) \quad = \quad \det(\lambda I_n - A) I_n.$$

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Let $A \in \mathbb{F}^{n \times n}$ be a matrix, and let $p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ be its characteristic polynomial. Then $A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n = O_{n\times n}$.

Proof. If $n = 1$, then the result is immediate. Indeed, suppose that $n = 1$, and consider any matrix $A = [a_{1,1}]$ in $\mathbb{F}^{1\times 1}$. Then $p_A(\lambda) = \det(\lambda I_1 - A) = \det(\left[\begin{array}{c} \lambda - a_{1,1} \end{array}\right]) = \lambda - a_{1,1}$, and we see that $A - a_{1,1}I_1 = O_{1\times 1}$.

So, assume that $n \geq 2$. By Theorem 1.1 applied to the matrix $\lambda I_n - A$ (where λ is a variable), we get that

$$(\lambda I_n - A)\,\text{adj}(\lambda I_n - A) \;\; = \;\; \det(\lambda I_n - A) I_n.$$

Now, note that each cofactor of the matrix $\lambda I_n - A$ is a polynomial (in variable λ) of degree at most λ^{n-1}, and consequently, each entry of $\text{adj}(\lambda I_n - A)$ is a polynomial (in the variable λ) of degree at most $n-1$.

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Proof (continued). So, the matrix $\mathrm{adj}(\lambda I_n - A)$ can be expressed in the form

$$\mathrm{adj}(\lambda I_n - A) = \lambda^{n-1}B_{n-1} + \lambda^{n-2}B_{n-2} + \cdots + \lambda B_1 + B_0,$$

for some matrices $B_0, B_1, \ldots, B_{n-1} \in \mathbb{F}^{n\times n}$.

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for some matrices $B_0, B_1, \dots, B_{n-1} \in \mathbb{F}^{n\times n}$. Consequently,

$$\underbrace{(\lambda I_n - A)(\underbrace{\lambda^{n-1}B_{n-1} + \lambda^{n-2}B_{n-2} + \cdots + \lambda B_1 + B_0}_{=\mathrm{adj}(\lambda I_n - A)})}_{:=\mathrm{LHS}} = \underbrace{\det(\lambda I_n - A)I_n}_{:=\mathrm{RHS}}.$$

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Proof (continued). For the left-hand-side, we have

$$\begin{aligned} \text{LHS} &= (\lambda I_n - A)(\lambda^{n-1}B_{n-1} + \cdots + \lambda B_1 + B_0) \\ &= \lambda^n B_{n-1} + \lambda^{n-1}(B_{n-2} - AB_{n-1}) + \lambda^{n-2}(B_{n-3} - AB_{n-2}) + \\ &\quad + \cdots + \lambda(B_0 - AB_1) - AB_0. \end{aligned}$$

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For the right-hand-side, we have

$$\begin{aligned}\text{RHS} &= \det(\lambda I_n - A)I_n \\ &= (\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-1} + \cdots + a_1\lambda + a_0)I_n \\ &= \lambda^n I_n + \lambda^{n-1}a_{n-1}I_n + \lambda^{n-2}a_{n-2}I_n + \cdots + \lambda a_1 I_n + a_0 I_n.\end{aligned}$$

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Proof (continued). The corresponding coefficients in front of λ^i (for $i \in \{0, 1, \dots, n\}$) must be equal on the left-hand-side (LHS) and the right-hand-side (RHS). This yields the following $n + 1$ equations.

$$\begin{array}{rcl} B_{n-1} & = & I_n \\ B_{n-2} - AB_{n-1} & = & a_{n-1} I_n \\ B_{n-3} - AB_{n-2} & = & a_{n-2} I_n \\ & \vdots & \\ B_0 - AB_1 & = & a_1 I_n \\ -AB_0 & = & a_0 I_n \end{array}$$

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We now multiply the first (top) equation by A^n on the left, the second equation by A^{n-1} on the left, the third equation by A^{n-2} on the left, and so on. This yields the following (next slide).

The Cayley-Hamilton theorem

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We now add up the equations that we obtained. On the left-hand-side, the sum is obviously $O_{n\times n}$.

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Proof (continued).

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We now add up the equations that we obtained. On the left-hand-side, the sum is obviously $O_{n\times n}$. So, the right-hand-side must also sum up to $O_{n\times n}$, i.e.

$$A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1 A + a_0 I_n = O_{n\times n}.$$

Q.E.D.

The Cayley-Hamilton theorem

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Corollary 1.2

For all matrices $A \in \mathbb{F}^{n\times n}$, both the following hold:

- (a) $A^n \in \text{Span}(I_n, A, A^2, \ldots, A^{n-1})$, i.e. A^n is a linear combination of the matrices $I_n, A, A^2, \ldots, A^{n-1}$;
- (b) if A is invertible, then $A^{-1} \in \text{Span}(I_n, A, A^2, \ldots, A^{n-1})$, i.e. A^{-1} is a linear combination of the matrices $I_n, A, A^2, \ldots, A^{n-1}$.

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Proof. Fix a matrix $A \in \mathbb{F}^{n \times n}$, and consider its characteristic polynomial $p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \dots + a_1\lambda + a_0$.

(a) By the Cayley-Hamilton theorem, we have that

$$A^n + a_{n-1}A^{n-1} + \dots + a_a A^2 + a_1 A + a_0 I_n \quad = \quad O_{n \times n}.$$

Consequently,

$$A^n \quad = \quad -a_0 I_n - a_1 A - a_2 A^2 - \dots - a_{n-1} A^{n-1}.$$

Thus, A^n is a linear combination of the matrices $I_n, A, A^2, \dots, A^{n-1}$, i.e. $A^n \in \text{Span}(I_n, A, A^2, \dots, A^{n-1})$.

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Now, by the Cayley-Hamilton theorem, we have that

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We multiply both sides of the equation by A^{-1} on the right, and we obtain $A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_2A + a_1I_n + a_0A^{-1} = O_{n\times n}$.

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Since $a_0 \neq 0$, this implies that

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2. Eigenvalues and eigenvectors of similar matrices

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- It is also easy to check that it is reflexive and transitive, and it follows that matrix similarity is an equivalence relation on $\mathbb{F}^{n\times n}$.
- By Theorem 4.3 of Lecture Notes 10, two matrices in $\mathbb{F}^{n\times n}$ are similar if and only if they represent the same linear transformation from an n-dimensional vector space V over $\mathbb{F}$ to itself, but possibly with respect to different bases of V.

Proposition 2.1

Let A and B be similar matrices in $\mathbb{F}^{n\times n}$. Then A and B have the same characteristic polynomial, as well as the same eigenvalues, with the same corresponding algebraic multiplicities, and with the same corresponding geometric multiplicities.

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Proof. Since A and B are similar, there exists an invertible matrix $P \in \mathbb{F}^{n\times n}$ such that $B = P^{-1}AP$. To see that A and B have the same characteristic polynomial, we compute:

$$\begin{aligned} p_B(\lambda) &= \det(\lambda I_n - B) \\ &= \det(\lambda I_n - P^{-1}AP) \\ &= \det(P^{-1}(\lambda I_n - A)P) \\ &= \underbrace{\det(P^{-1})}_{=\frac{1}{\det(P)}} \underbrace{\det(\lambda I_n - A)}_{=p_A(\lambda)} \det(P) \quad = \quad p_A(\lambda). \end{aligned}$$

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Now, fix an eigenvalue λ of A and B, and let m_A and m_B be the geometric multiplicities of λ as an eigenvalue of A and B, respectively. WTS $m_A = m_B$. Since matrix similarity is symmetric, it suffices to show that $m_A \leq m_B$. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_{m_A}\}$ be a basis for the eigenspace $E_\lambda(A)$.

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Let A and B be similar matrices in $\mathbb{F}^{n\times n}$. Then A and B have the same characteristic polynomial, as well as the same eigenvalues, with the same corresponding algebraic multiplicities, and with the same corresponding geometric multiplicities.

Proof (continued). So, A and B have the same characteristic polynomial, and consequently, the same eigenvalues with the same corresponding algebraic multiplicities.

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Proof (continued).

$$\begin{aligned}
B(P^{-1}\mathbf{v}_i) &= (\underbrace{P^{-1}AP}_{=B})(P^{-1}\mathbf{v}_i) \\
&= P^{-1}A(\underbrace{PP^{-1}}_{=I_n})\mathbf{v}_i \\
&= P^{-1}(A\mathbf{v}_i) \\
&= P^{-1}(\lambda\mathbf{v}_i) && \text{because } \mathbf{v}_i \in E_\lambda(A) \\
&= \lambda(P^{-1}\mathbf{v}_i).
\end{aligned}$$

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Proof (continued). Reminder: $P^{-1}\mathbf{v}_1, \dots, P^{-1}\mathbf{v}_{m_A}$ are eigenvectors of B associated with the eigenvalue λ_i.

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Proposition 2.2

Let $B, C \in \mathbb{F}^{n \times n}$ be similar matrices, with $C = P^{-1}BP$ for some invertible matrix $P \in \mathbb{F}^{n \times n}$. Then for all non-negative integers m, we have that $C^m = P^{-1}B^mP$ and $B^m = PC^mP^{-1}$.

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Let us prove the first one. We proceed by induction on m. First, we have that $C^0 = I_n$ and that $P^{-1}B^0P = P^{-1}I_nP = I_n$, and so $C^0 = P^{-1}B^0P$. Thus, the claim holds for $m = 0$.

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Proof (continued). Now, fix a non-negative integer m, and assume inductively that $C^m = P^{-1}B^mP$. Then

$$\begin{aligned} C^{m+1} &= C^mC \\ &= (\underbrace{P^{-1}B^mP}_{=C^m})(\underbrace{P^{-1}BP}_{=C}) \qquad \text{by the induction hypothesis} \\ &= P^{-1}B^m(\underbrace{PP^{-1}}_{=I_n})BP \\ &= P^{-1}B^{m+1}P. \end{aligned}$$

This completes the induction. Q.E.D.

3. Algebraic and geometric multiplicities revisited

Theorem 2.5 of Lecture Notes 18

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Then the geometric multiplicity of any eigenvalue of A is no greater than the algebraic multiplicity of that eigenvalue.

Proof.

3. Algebraic and geometric multiplicities revisited

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Proof. Suppose that λ_0 is an eigenvalue of A of geometric multiplicity k. WTS the eigenvalue λ_0 has algebraic multiplicity at least k. In view of Proposition 2.1, it is enough to exhibit a matrix $B \in \mathbb{F}^{n\times n}$ similar to A, and such that λ_0 is an eigenvalue of B of algebraic multiplicity at least k.

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Since the geometric multiplicity of the eigenvalue λ_0 of A is k, we see that the eigenspace E_{λ_0} has a k-element basis, say $\{\mathbf{p}_1, \dots, \mathbf{p}_k\}$.

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Since the geometric multiplicity of the eigenvalue λ_0 of A is k, we see that the eigenspace E_{λ_0} has a k-element basis, say $\{\mathbf{p}_1, \dots, \mathbf{p}_k\}$. We now extend $\{\mathbf{p}_1, \dots, \mathbf{p}_k\}$ to a basis $\mathcal{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_k, \mathbf{p}_{k+1}, \dots, \mathbf{p}_n\}$ of $\mathbb{F}^n$.

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Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Then the geometric multiplicity of any eigenvalue of A is no greater than the algebraic multiplicity of that eigenvalue.

Proof (continued). Reminder: $\{\mathbf{p}_1, \dots, \mathbf{p}_k\}$ is a basis of E_{λ_0}; $\mathcal{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_k, \mathbf{p}_{k+1}, \dots, \mathbf{p}_n\}$ of $\mathbb{F}^n$.

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Proof (continued). Reminder: $\{\mathbf{p}_1, \dots, \mathbf{p}_k\}$ is a basis of E_{λ_0}; $\mathcal{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_k, \mathbf{p}_{k+1}, \dots, \mathbf{p}_n\}$ of $\mathbb{F}^n$.

Let $f_A : \mathbb{F}^n \to \mathbb{F}^n$ be the linear transformation whose standard matrix is A (i.e. $f_A(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{F}^n$), so that $A = {}_{\mathcal{E}_n}[f_A]_{\mathcal{E}_n}$, where $\mathcal{E}_n = \{\mathbf{e}_1^n, \dots, \mathbf{e}_n^n\}$ is the standard basis of $\mathbb{F}^n$. Consider the matrix $B := {}_{\mathcal{P}}[f_A]_{\mathcal{P}}$. Then A and B are similar, since (next slide):

Theorem 2.5 of Lecture Notes 18

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Then the geometric multiplicity of any eigenvalue of A is no greater than the algebraic multiplicity of that eigenvalue.

Proof (continued). Reminder: $\{\mathbf{p}_1, \dots, \mathbf{p}_k\}$ is a basis of E_{λ_0}; $\mathcal{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_k, \mathbf{p}_{k+1}, \dots, \mathbf{p}_n\}$ of $\mathbb{F}^n$.

$$\begin{aligned} A &= {}_{\mathcal{E}_n}[f_A]_{\mathcal{E}_n} \\ &= {}_{\mathcal{E}_n}[Id_{\mathbb{F}^n} \circ f_A \circ Id_{\mathbb{F}^n}]_{\mathcal{E}_n} \\ &= {}_{\mathcal{E}_n}[Id_{\mathbb{F}^n}]_{\mathcal{P}} \; {}_{\mathcal{P}}[f_A]_{\mathcal{P}} \; {}_{\mathcal{P}}[Id_{\mathbb{F}^n}]_{\mathcal{E}_n} \\ &= ({}_{\mathcal{P}}[Id_{\mathbb{F}^n}]_{\mathcal{E}_n})^{-1} \; \underbrace{{}_{\mathcal{P}}[f_A]_{\mathcal{P}}}_{=B} \; {}_{\mathcal{P}}[Id_{\mathbb{F}^n}]_{\mathcal{E}_n}, \end{aligned}$$

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$$\begin{aligned}
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&= {}_{\mathcal{E}_n}[Id_{\mathbb{F}^n} \circ f_A \circ Id_{\mathbb{F}^n}]_{\mathcal{E}_n} \\
&= {}_{\mathcal{E}_n}[Id_{\mathbb{F}^n}]_{\mathcal{P}} \ {}_{\mathcal{P}}[f_A]_{\mathcal{P}} \ {}_{\mathcal{P}}[Id_{\mathbb{F}^n}]_{\mathcal{E}_n} \\
&= ({}_{\mathcal{P}}[Id_{\mathbb{F}^n}]_{\mathcal{E}_n})^{-1} \ \underbrace{{}_{\mathcal{P}}[f_A]_{\mathcal{P}}}_{=B} \ {}_{\mathcal{P}}[Id_{\mathbb{F}^n}]_{\mathcal{E}_n},
\end{aligned}$$

(Alternatively, the similarity of A and B follows from Theorem 4.3 of Lecture Notes 10.)

Proof (continued). It now remains to show that λ_0 is an eigenvalue of B of algebraic multiplicity at least k.

Proof (continued). It now remains to show that λ_0 is an eigenvalue of B of algebraic multiplicity at least k. For this, we observe that

$$\begin{aligned}
B &= {}_{\mathcal{P}}[f_A]_{\mathcal{P}} \\
&= \begin{bmatrix} [f_A(\mathbf{p}_1)]_{\mathcal{P}} & \dots & [f_A(\mathbf{p}_k)]_{\mathcal{P}} & [f_A(\mathbf{p}_{k+1})]_{\mathcal{P}} & \dots & [f_A(\mathbf{p}_n)]_{\mathcal{P}} \end{bmatrix} \\
&= \begin{bmatrix} [A\mathbf{p}_1]_{\mathcal{P}} & \dots & [A\mathbf{p}_k]_{\mathcal{P}} & [f_A(\mathbf{p}_{k+1})]_{\mathcal{P}} & \dots & [f_A(\mathbf{p}_n)]_{\mathcal{P}} \end{bmatrix} \\
&= \begin{bmatrix} [\lambda_0\mathbf{p}_1]_{\mathcal{P}} & \dots & [\lambda_0\mathbf{p}_k]_{\mathcal{P}} & [f_A(\mathbf{p}_{k+1})]_{\mathcal{P}} & \dots & [f_A(\mathbf{p}_n)]_{\mathcal{P}} \end{bmatrix} \\
&= \begin{bmatrix} \lambda_0\mathbf{e}_1^n & \dots & \lambda_0\mathbf{e}_k^n & [f_A(\mathbf{p}_{k+1})]_{\mathcal{P}} & \dots & [f_A(\mathbf{p}_n)]_{\mathcal{P}} \end{bmatrix} \\
&= \left[\begin{array}{c|ccc} \lambda_0 I_k & & & \\ \hline O_{(n-k)\times k} & [f_A(\mathbf{p}_{k+1})]_{\mathcal{P}} & \dots & [f_A(\mathbf{p}_n)]_{\mathcal{P}} \end{array} \right].
\end{aligned}$$

Proof (continued). We now have that

$$p_B(\lambda) \;=\; \det(\lambda I_n - B) \;=\; \left| \begin{array}{c|c} (\lambda - \lambda_0) I_k & \\ \hline O_{(n-k)\times k} & C \end{array} \right|,$$

where

$$C \;=\; \left[\begin{array}{ccc} \lambda \mathbf{e}_{k+1} - [f_A(\mathbf{p}_{k+1})]_{\mathcal{P}} & \ldots & \lambda \mathbf{e}_n - [f_A(\mathbf{p}_{k+1})]_{\mathcal{P}} \end{array} \right]_{(n-k)\times n}.$$

Proof (continued). We now have that

$$p_B(\lambda) = \det(\lambda I_n - B) = \left| \begin{array}{c|c} (\lambda - \lambda_0) I_k & C \\ \hline O_{(n-k)\times k} & \end{array} \right|,$$

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$$C = \begin{bmatrix} \lambda \mathbf{e}_{k+1} - [f_A(\mathbf{p}_{k+1})]_{\mathcal{P}} & \ldots & \lambda \mathbf{e}_n - [f_A(\mathbf{p}_{k+1})]_{\mathcal{P}} \end{bmatrix}_{(n-k)\times n}.$$

Thus, $p_B(\lambda)$ has the form

$$p_B(\lambda) = \left| \begin{array}{cccc|cccc} \lambda - \lambda_0 & 0 & \ldots & 0 & * & * & \ldots & * \\ 0 & \lambda - \lambda_0 & \ldots & 0 & * & * & \ldots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda - \lambda_0 & * & * & \ldots & * \\ \hline 0 & 0 & \ldots & 0 & * & * & \ldots & * \\ 0 & 0 & \ldots & 0 & * & * & \ldots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & * & * & \ldots & * \end{array} \right|,$$

where the red matrix in the upper-left corner is of size $k \times k$.

Proof (continued). Reminder:

$$p_B(\lambda) = \left|\begin{array}{cccc:cccc} \lambda-\lambda_0 & 0 & \dots & 0 & * & * & \dots & * \\ 0 & \lambda-\lambda_0 & \dots & 0 & * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda-\lambda_0 & * & * & \dots & * \\ \hdashline 0 & 0 & \dots & 0 & * & * & \dots & * \\ 0 & 0 & \dots & 0 & * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & * & * & \dots & * \end{array}\right|,$$

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Proof (continued). Reminder:

$$p_B(\lambda) \quad = \quad \left| \begin{array}{cccc:cccc} \lambda-\lambda_0 & 0 & \dots & 0 & * & * & \dots & * \\ 0 & \lambda-\lambda_0 & \dots & 0 & * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda-\lambda_0 & * & * & \dots & * \\ \hdashline 0 & 0 & \dots & 0 & * & * & \dots & * \\ 0 & 0 & \dots & 0 & * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & * & * & \dots & * \end{array} \right|,$$

where the red matrix in the upper-left corner is of size $k \times k$.

By iteratively performing Laplace expansion along the first column, we see that $p_B(\lambda)$ has a factor $(\lambda - \lambda_0)^k$, and consequently, λ_0 is an eigenvalue of B of algebraic multiplicity at least k. Q.E.D.

Theorem 2.5 of Lecture Notes 18

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$. Then the geometric multiplicity of any eigenvalue of A is no greater than the algebraic multiplicity of that eigenvalue.

4. Diagonal matrices and diagonalization

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- A matrix $A = [\ a_{i,j}\ ]_{n\times n}$ in $\mathbb{F}^{n\times n}$ is *diagonal* if $a_{i,j} = 0$ for all distinct $i, j \in \{1, \ldots, n\}$.

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4. Diagonal matrices and diagonalization

- A matrix $A = [\ a_{i,j}\ ]_{n\times n}$ in $\mathbb{F}^{n\times n}$ is *diagonal* if $a_{i,j} = 0$ for all distinct $i, j \in \{1, \dots, n\}$.
    - In other words, a square matrix is diagonal if all its entries off the main diagonal are zero (and the main diagonal is arbitrary).
- Given scalars $\lambda_1, \dots, \lambda_n \in \mathbb{F}$, we will denote by $D(\lambda_1, \dots, \lambda_n)$ the diagonal matrix in which $\lambda_1, \dots, \lambda_n$ appear on the main diagonal (in that order), i.e.

$$
\begin{aligned}
D(\lambda_1, \lambda_2, \dots, \lambda_n) \quad &:= \quad \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \\
&= \quad \begin{bmatrix} \lambda_1\mathbf{e}_1 & \lambda_2\mathbf{e}_2 & \dots & \lambda_n\mathbf{e}_n \end{bmatrix}.
\end{aligned}
$$

Porposition 4.1

Let $\lambda_1, \dots, \lambda_n \in \mathbb{F}^n$, and set $D := D(\lambda_1, \dots, \lambda_n)$. Then both the following hold:

(a) For all matrices $P = \left[\begin{array}{ccc} \mathbf{p}_1 & \dots \mathbf{p}_n \end{array} \right]$ in $\mathbb{F}^{n \times n}$, we have that $PD = \left[\begin{array}{ccc} \lambda_1 \mathbf{p}_1 & \dots \lambda_n \mathbf{p}_n \end{array} \right]$;

(b) For all non-negative integers m, we have that $D^m = D(\lambda_1^m, \dots, \lambda_n^m)$.

Proof.

Porposition 4.1

Let $\lambda_1, \dots, \lambda_n \in \mathbb{F}^n$, and set $D := D(\lambda_1, \dots, \lambda_n)$. Then both the following hold:

- (a) For all matrices $P = \begin{bmatrix} \mathbf{p}_1 & \dots \mathbf{p}_n \end{bmatrix}$ in $\mathbb{F}^{n \times n}$, we have that $PD = \begin{bmatrix} \lambda_1 \mathbf{p}_1 & \dots \lambda_n \mathbf{p}_n \end{bmatrix}$;
- (b) For all non-negative integers m, we have that $D^m = D(\lambda_1^m, \dots, \lambda_n^m)$.

Proof. (a) Fix a matrix $P = \begin{bmatrix} \mathbf{p}_1 & \dots & \mathbf{p}_n \end{bmatrix}$ in $\mathbb{F}^{n \times n}$. Then

$$\begin{aligned} PD &= P \begin{bmatrix} \lambda_1 \mathbf{e}_1 & \dots & \lambda_n \mathbf{e}_n \end{bmatrix} \\ &= \begin{bmatrix} P(\lambda_1 \mathbf{e}_1) & \dots & P(\lambda_n \mathbf{e}_n) \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 (P\mathbf{e}_1) & \dots & \lambda_n (P\mathbf{e}_n) \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 \mathbf{p}_1 & \dots & \lambda_n \mathbf{p}_n \end{bmatrix}. \end{aligned}$$

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(b) For all non-negative integers m, we have that $D^m = D(\lambda_1^m, \dots, \lambda_n^m)$.

Proof (continued). (b) readily follows from (a) via an easy induction on m. (Details: Lecture Notes.)

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- We will prove Theorem 2.4.
- First, however, it may be useful to recall when $\mathbb{F}^n$ has a basis formed by eigenvectors of $A^{n\times n}$.
- This is precisely described in Corollary 2.13 of Lecture Notes 18.

Corollary 2.13 of Lecture Notes 18

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Then the following are equivalent:

(i) $\mathbb{F}^n$ has a basis formed by eigenvectors of A;

(ii) the sum of algebraic multiplicities of all distinct eigenvalues A is equal to n, and the geometric multiplicity of each eigenvalue of A is equal to the algebraic multiplicity of that eigenvalue;

(iii) the sum of geometric multiplicities of all distinct eigenvalues A is equal to n.

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A matrix $A \in \mathbb{F}^{n \times n}$ is diagonalizable if and only if $\mathbb{F}^n$ has a basis formed by eigenvectors of A.

- Let's prove Theorem 4.2!

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Proof.

Theorem 4.2

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Proof. Fix a matrix $A \in \mathbb{F}^{n\times n}$.

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A matrix $A \in \mathbb{F}^{n \times n}$ is diagonalizable if and only if $\mathbb{F}^n$ has a basis formed by eigenvectors of A.

Proof. Fix a matrix $A \in \mathbb{F}^{n \times n}$. Suppose first that A is diagonalizable, and fix matrices $P, D \in \mathbb{F}^{n \times n}$ such that D is diagonal, P is invertible, and $D = P^{-1}AP$.

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A matrix $A \in \mathbb{F}^{n \times n}$ is diagonalizable if and only if $\mathbb{F}^n$ has a basis formed by eigenvectors of A.

Proof (continued). Suppose now that $\mathbb{F}^n$ has a basis, say, $\mathcal{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$, formed by eigenvectors of A.

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For each $i \in \{1, \dots, n\}$, let λ_i be the eigenvalue of A associated with the eigenvector $\mathbf{p}_i$, so that $A\mathbf{p}_i = \lambda_i \mathbf{p}_i$. Set $D = D(\lambda_1, \dots, \lambda_n)$ and $P = \begin{bmatrix} \mathbf{p}_1 & \dots & \mathbf{p}_n \end{bmatrix}$.

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Note that for each $i \in \{1, \dots, n\}$, the i-th column of AP is $A\mathbf{p}_i$, the i-th column of PD is $\lambda_i \mathbf{p}_i$ (by Proposition 4.1(a)), and we know that $A\mathbf{p}_i = \lambda_i \mathbf{p}_i$.

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Theorem 4.2

A matrix $A \in \mathbb{F}^{n \times n}$ is diagonalizable if and only if $\mathbb{F}^n$ has a basis formed by eigenvectors of A.

- If $D = P^{-1}AP$ for some diagonal matrix $D = D(\lambda_1, \ldots, \lambda_n)$ and invertible matrix $P = [\ \mathbf{p}_1 \quad \ldots \quad \mathbf{p}_n\ ]$, then A and D are the matrices of the same linear transformation, but with respect to different bases.

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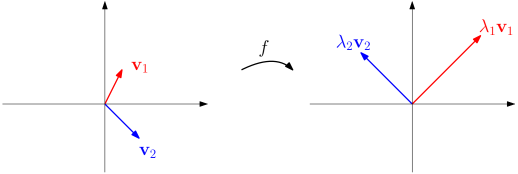
- Consider the unique linear transformation $f : \mathbb{F}^n \to \mathbb{F}^n$ such that $f(\mathbf{p}_1) = \lambda_1\mathbf{p}_1, \ldots, f(\mathbf{p}_n) = \lambda_n\mathbf{p}_n$.



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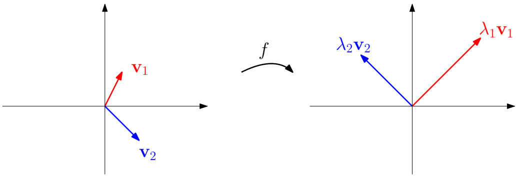
- Consider the unique linear transformation $f : \mathbb{F}^n \to \mathbb{F}^n$ such that $f(\mathbf{p}_1) = \lambda_1\mathbf{p}_1, \ldots, f(\mathbf{p}_n) = \lambda_n\mathbf{p}_n$.



- Then $A = {}_{\mathcal{E}_n}[f]_{\mathcal{E}_n}$ and $D = {}_{\mathcal{P}}[f]_{\mathcal{P}}$, where $\mathcal{E}_n$ is the standard basis of $\mathbb{F}^n$ and $\mathcal{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$.

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Let $A \in \mathbb{F}^{n \times n}$ be a matrix that has n distinct eigenvalues. Then A is diagonalizable.

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Proof. Since A has n distinct eigenvalues, and each of those eigenvalues has geometric multiplicity at least 1 (by the definition of an eigenvalue), we see that the sum of geometric multiplicities of the distinct eigenvalues of A is at least n.

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Proof. Since A has n distinct eigenvalues, and each of those eigenvalues has geometric multiplicity at least 1 (by the definition of an eigenvalue), we see that the sum of geometric multiplicities of the distinct eigenvalues of A is at least n. But then the sum of geometric multiplicities of the distinct eigenvalues of A must be exactly n.

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- The theory we have developed so far (including both the statement and the proof of Theorem 4.2) gives us a recipe for determining whether a matrix $A \in \mathbb{F}^{n\times n}$ is diagonalizable, and if so, diagonalizing it.
- We proceed as follows (the steps are in numerals, and justifications of individual steps are given in bullet points underneath the steps in question).

1. We compute the characteristic polynomial $p_A(\lambda)$, factor it, and compute all the eigenvalues of A together with their algebraic multiplicities.
2. If the algebraic multiplicities of the eigenvalues of A add up to less than n, then A is not diagonalizable.
    - **Note:** The sum of algebraic multiplicities of the eigenvalues of A will always be at most n, and if the field $\mathbb{F}$ is algebraically closed (for example, if $\mathbb{F} = \mathbb{C}$), then it will be equal to n. If $\mathbb{F}$ is **not** algebraically closed (for example, if $\mathbb{F}$ is $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{Z}_p$ for some prime number p), then it is possible that the sum of algebraic multiplicities of the eigenvalues of A is less than n. In this case, Corollary 2.13 of Lecture Notes 18 guarantees that $\mathbb{F}$ does not have a basis formed by eigenvectors of A, and consequently (by Theorem 4.2), A is not diagonalizable.

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3. From now on, assume that the algebraic multiplicities of the eigenvalues of A add up to n. For each eigenvalue λ of A, find a basis of the eigenspace E_λ and the geometric multiplicity of λ.

4. If the geometric multiplicity of some eigenvalue of A is less than its algebraic multiplicity, then A is not diagonalizable.
    - **Note:** If the geometric multiplicity of some eigenvalue of A is less than its algebraic multiplicity, then Corollary 2.13 of Lecture Notes 18 guarantees that $\mathbb{F}^n$ has no basis formed by eigenvectors of A, and so by Theorem 4.2, A is not diagonalizable.

4. If the geometric multiplicity of some eigenvalue of A is less than its algebraic multiplicity, then A is not diagonalizable.
   - **Note:** If the geometric multiplicity of some eigenvalue of A is less than its algebraic multiplicity, then Corollary 2.13 of Lecture Notes 18 guarantees that $\mathbb{F}^n$ has no basis formed by eigenvectors of A, and so by Theorem 4.2, A is not diagonalizable.
5. From now on, assume that the geometric multiplicity of each eigenvalue of A is equal to its algebraic multiplicity, so that the geometric multiplicities of the eigenvalues of A add up to n. In this case, A is diagonalizable.
   - **Note:** Indeed, if the geometric multiplicities of the eigenvectors of A add up to n, then Corollary 2.13 of Lecture Notes 18 guarantees that $\mathbb{F}^n$ has a basis formed by eigenvectors of A, and so by Theorem 4.2, A is diagonalizable.

6. Let $\lambda_1, \dots, \lambda_\ell$ be the distinct eigenvalues of A, with geometric multiplicities $n_1, \dots, n_\ell$, respectively. (So, $n_1 + \dots + n_\ell = n$.) We now form the diagonal matrix
$D = D(\underbrace{\lambda_1, \dots, \lambda_1}_{n_1}, \dots, \underbrace{\lambda_\ell, \dots, \lambda_\ell}_{n_\ell})$, and we form the matrix P
as follows: the first n_1 columns of P form a basis of the eigenspace E_{λ_1}, the next n_2 columns of P form a basis of the eigenspace E_{λ_2}, and so on, until the last n_ℓ columns of P form a basis of the eigenspace E_{λ_ℓ}. Now $D = PAP^{-1}$

   - **Note:** Theorem 2.12 of Lecture Notes 18 guarantees that the columns of P form a linearly independent set; since P has n columns, it follows that the columns of P form a basis of $\mathbb{F}^n$. Now the correctness of our construction follows from the proof of Theorem 4.2.

Example 4.4

Consider the matrix

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

in $\mathbb{C}^{3\times 3}$. Determine whether A is diagonalizable, and if so, diagonalize it.

Solution.

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in $\mathbb{C}^{3\times 3}$. Determine whether A is diagonalizable, and if so, diagonalize it.

Solution. The matrix A is precisely the matrix from Example 2.6 of Lecture Notes 18. In that example, we determined that A has two eigenvalues, namely, $\lambda_1 = 4$ (with algebraic multiplicity 1 and geometric multiplicity 1) and $\lambda_2 = 5$ (with algebraic multiplicity 2 and geometric multiplicity 2). Since the sum of geometric multiplicities is 3, we see that the 3×3 matrix A is indeed diagonalizable.

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Solution (continued). In Example 2.6 of Lecture Notes 18, we saw that $\{[\ -1 \quad 2 \quad 0\ ]^T\}$ is a basis of the eigespace E_{λ_1}, and that $\{[\ 0 \quad 1 \quad 0\ ]^T, [\ -2 \quad 0 \quad 1\ ]^T\}$ is a basis of the eigenspace E_{λ_2}.

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$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

in $\mathbb{C}^{3\times 3}$. Determine whether A is diagonalizable, and if so, diagonalize it.

Solution (continued). In Example 2.6 of Lecture Notes 18, we saw that $\{[\begin{matrix} -1 & 2 & 0 \end{matrix}]^T\}$ is a basis of the eigespace E_{λ_1}, and that $\{[\begin{matrix} 0 & 1 & 0 \end{matrix}]^T, [\begin{matrix} -2 & 0 & 1 \end{matrix}]^T\}$ is a basis of the eigenspace E_{λ_2}. So, we set

$$D := \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad P := \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and we see that $D = P^{-1}AP$.

Example 4.5

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

in $\mathbb{C}^{5\times 5}$. Determine whether A is diagonalizable, and if so, diagonalize it.

Solution

Example 4.5

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

in $\mathbb{C}^{5\times 5}$. Determine whether A is diagonalizable, and if so, diagonalize it.

Solution The matrix A is precisely the matrix from Example 2.10 of Lecture Notes 18. In that example, we determined that A has three eigenvalues, namely $\lambda_1 = 1$ (with algebraic multiplicity 2 and geometric multiplicity 2), $\lambda_2 = 2$ (with algebraic multiplicity 1 and geometric multiplicity 1), and $\lambda_3 = 3$ (with algebraic multiplicity 2 and geometric multiplicity 1). Since the geometric multiplicity of the eigenvalue $\lambda_3 = 3$ is strictly smaller than the algebraic multiplicity, we see that A is not diagonalizable.

- One reason we care about diagonalizability is because diagonalizable matrices are easy to exponentiate.

- Suppose, furthermore, that we have diagonalized A as $D = P^{-1}AP$, where $D = D(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix and P an invertible matrix in $\mathbb{F}^{n \times n}$.

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- Indeed, suppose we are given a diagonalizable matrix $A \in \mathbb{F}^{n \times n}$.
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- On the other hand, by Proposition 4.1(b), we have that $D^m = D(\lambda_1^m, \ldots, \lambda_n^m)$.
- So, A^m is easy to compute.

Example 4.6

Consider the matrix

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

in $\mathbb{C}^{3\times 3}$. Find a formula for A^m (where m is an arbitrary non-negative integer).

Solution.

Example 4.6

Consider the matrix

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

in $\mathbb{C}^{3\times 3}$. Find a formula for A^m (where m is an arbitrary non-negative integer).

Solution. This is the matrix from Example 4.4.

Example 4.6

Consider the matrix

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

in $\mathbb{C}^{3\times 3}$. Find a formula for A^m (where m is an arbitrary non-negative integer).

Solution. This is the matrix from Example 4.4. In that example, we computed matrices $D, P \in \mathbb{C}^{3\times 3}$ such that D is diagonal, P is invertible, and $D = P^{-1}AP$. The matrices in question were

$$D := \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad P := \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 4.6

Consider the matrix

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

in $\mathbb{C}^{3\times 3}$. Find a formula for A^m (where m is an arbitrary non-negative integer).

Solution (continued). We then compute

$$P^{-1} = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution (continued). Then for all non-negative integers m, we have the following:

$$\begin{aligned} A^m &\overset{(*)}{=} PD^mP^{-1} \\ &\overset{(**)}{=} \underbrace{\begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=P} \underbrace{\begin{bmatrix} 4^m & 0 & 0 \\ 0 & 5^m & 0 \\ 0 & 0 & 5^m \end{bmatrix}}_{=D^m} \underbrace{\begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}}_{=P^{-1}} \\ &\overset{(***)}{=} \begin{bmatrix} 4^m & 0 & 2\cdot 4^m - 2\cdot 5^m \\ -2\cdot 4^m + 2\cdot 5^m & 5^m & -4^{m+1} + 4\cdot 5^m \\ 0 & 0 & 5^m \end{bmatrix} \\ &= \begin{bmatrix} 4^m & 0 & 2(4^m - 5^m) \\ 2(5^m - 4^m) & 5^m & 4(5^m - 4^m) \\ 0 & 0 & 5^m \end{bmatrix}, \end{aligned}$$

where (*) follows from Proposition 2.2, (**) follows from Proposition 4.1, and (***) follows by simple matrix multiplication.

Example 4.6

Consider the matrix

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

in $\mathbb{C}^{3\times 3}$. Find a formula for A^m (where m is an arbitrary non-negative integer).

Solution (continued). Reminder:

$$A^m = \begin{bmatrix} 4^m & 0 & 2(4^m - 5^m) \\ 2(5^m - 4^m) & 5^m & 4(5^m - 4^m) \\ 0 & 0 & 5^m \end{bmatrix}$$

for all non-negative integers m.

Example 4.6

Consider the matrix

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

in $\mathbb{C}^{3\times 3}$. Find a formula for A^m (where m is an arbitrary non-negative integer).

Solution (continued). Reminder:

$$A^m = \begin{bmatrix} 4^m & 0 & 2(4^m - 5^m) \\ 2(5^m - 4^m) & 5^m & 4(5^m - 4^m) \\ 0 & 0 & 5^m \end{bmatrix}$$

for all non-negative integers m.

Optionally, we can verify by induction on m that our formula is correct (details: Lecture Notes).