Linear Algebra 2: Lecture 19

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Note: Throughout this lecture, $\mathbb{F}$ is a fixed field.

1 The Cayley-Hamilton theorem

The famous Cayley-Hamilton theorem essentially states that every square matrix is a root of its own characteristic polynomial. (Here, we need to treat the free coefficient of the characteristic polynomial as that coefficient times the identity matrix of the appropriate size.) For example, for a matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

we have

$$p_A(\lambda) = \det(\lambda I_2 - A) = \begin{vmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{vmatrix} = \lambda^2 - 5\lambda - 2,$$

and

$$\begin{aligned} A^2 - 5A - 2I_2 &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 - 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Before proving the Cayley-Hamilton theorem, we first need to revisit the adjugate matrix. Recall that for a matrix $A = \left[\ a_{i,j}\ \right]_{n\times n}$ ($n \geq 2$) in $\mathbb{F}^{n\times n}$ and indices $i, j \in \{1, \dots, n\}$, $A_{i,j}$ is the $(n-1)\times(n-1)$ matrix obtained from A by deleting the i-th row and j-th column, and the cofactor $C_{i,j}$ is defined to be $C_{i,j} := (-1)^{i+j}\det(A_{i,j})$. The *cofactor matrix* of A is the matrix $\left[\begin{array}{c} C_{i,j} \end{array}\right]_{n\times n}$. The *adjugate matrix* (also called the *classical adjoint*) of A, denoted by $\mathrm{adj}(A)$, is the transpose of the cofactor matrix of A, i.e.

$$\mathrm{adj}(A) \quad := \quad \left[\begin{array}{c} C_{i,j} \end{array}\right]_{n\times n}{}^{T}.$$

We will use Theorem 1.1 (below) in our proof of the Cayley-Hamilton theorem. We note that the second statement of Theorem 1.1 is precisely Theorem 4.2 of Lecture Notes 16. (So, Theorem 1.1 of the present lecture is stronger than Theorem 4.2 of Lecture Notes 16. We note that Theorem 4.2 of Lecture Notes 16 was obtained as a corollary of Cramer's rule, which in fact cannot be used - at least not directly - to prove the first statement of Theorem 1.1 below.) However, the proof of the Cayley-Hamilton theorem uses the first statement of Theorem 1.1.

Theorem 1.1. *Let* $A \in \mathbb{F}^{n\times n}$ *(*$n \geq 2$*). Then*

$$\mathit{adj}(A)\ A \quad = \quad \mathit{det}(A) I_n \quad = \quad A\ \mathit{adj}(A).$$

Consequently, if A *is invertible, then* $A^{-1} = \frac{1}{\mathit{det}(A)}\mathit{adj}(A)$.

Proof. Let us first show that the first statement implies the second. Indeed, if A is invertible, then $\det(A) \neq 0$, and so if the first statement holds, then we get that

$$\left(\tfrac{1}{\det(A)}\mathrm{adj}(A)\right) A \quad = \quad I_n \quad = \quad A \left(\tfrac{1}{\det(A)}\mathrm{adj}(A)\right),$$

and consequently, $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$.

It remains to prove the first statement, i.e. that $\mathrm{adj}(A)\ A = \det(A) I_n = A\ \mathrm{adj}(A)$. We will do this by proving that the matrices $\mathrm{adj}(A)\ A$, $\det(A) I_n$, and $A\ \mathrm{adj}(A)$ have the same corresponding entries. Fix indices $i, j \in \{1, \dots, n\}$. The i, j-th entry of the matrix $\det(A) I_n$ is $\det(A)$ if $i = j$, and is zero if $i \neq j$. We must show this holds for the i, j-th entry of the matrices $\mathrm{adj}(A)\ A$ and $A\ \mathrm{adj}(A)$ as well.

We first consider the matrix $\mathrm{adj}(A)\ A$. The i-th row of $\mathrm{adj}(A)$ is $\left[\begin{array}{ccc} C_{1,i} & \dots & C_{n,i} \end{array}\right]$, and the j-th column of A is $\left[\begin{array}{ccc} a_{1,j} & \dots & a_{n,j} \end{array}\right]^T$. So, the i, j-th entry of $\mathrm{adj}(A)\ A$ is $\sum_{k=1}^{n} a_{k,j} C_{k,i}$. Now, let B_1 be the matrix obtained by replacing the i-th column of A by the j-th column of A. Then $\det(B_1) = \sum_{k=1}^{n} a_{k,j} C_{k,i}$ (via Laplace expansion along the i-th column of B_1). But if $i = j$, then $\det(B_1) = \det(A)$ (because $B_1 = A$), and if $i \neq j$, then $\det(B_1) = 0$ (because B_1 has two identical columns, namely, the i-th and j-th column).[1]

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[1] By Proposition 2.4 of Lecture Notes 15, the determinant of a square matrix with two identical columns is zero.

We now consider the matrix $A\,\mathrm{adj}(A)$. The i-th row of the matrix A is $\begin{bmatrix} a_{i,1} & \dots & a_{i,n} \end{bmatrix}$ and the j-th column of $\mathrm{adj}(A)$ is $\begin{bmatrix} C_{j,1} & \dots & C_{j,n} \end{bmatrix}^T$. So, the i,j-th entry of $A\,\mathrm{adj}(A)$ is $\sum_{k=1}^{n} a_{i,k}C_{j,k}$. Now, let B_2 be the matrix obtained by replacing the j-th row of A by the i-th row of A. Then $\det(B_2) = \sum_{k=1}^{n} a_{i,k}C_{j,k}$ (via Laplace expansion along the j-th row of B_2). But if $i = j$, then $\det(B_2) = \det(A)$ (because $B_2 = A$), and if $i \neq j$, then $\det(B_2) = 0$ (because B_2 has two identical rows, namely, the i-th and j-th row).[2] ☐

The Cayley-Hamilton theorem. *Let $A \in \mathbb{F}^{n\times n}$ be a matrix, and let $p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$ be its characteristic polynomial. Then*

$$A^n + a_{n-1}A^{n-1} + \dots + a_1A + a_0I_n \quad = \quad O_{n\times n}.$$

Proof. If $n = 1$, then the result is immediate.[3] So, assume that $n \geq 2$. By Theorem 1.1 applied to the matrix $\lambda I_n - A$ (where λ is a variable), we get that

$$(\lambda I_n - A)\,\mathrm{adj}(\lambda I_n - A) \quad = \quad \det(\lambda I_n - A)I_n.$$

Now, note that each cofactor of the matrix $\lambda I_n - A$ is a polynomial (in variable λ) of degree at most λ^{n-1}. Since the entries of $\mathrm{adj}(\lambda I_n - A)$ are precisely the cofactors of $\lambda I_n - A$, it follows that each entry of $\mathrm{adj}(\lambda I_n - A)$ is a polynomial (in the variable λ) of degree at most $n-1$. So, the matrix $\mathrm{adj}(\lambda I_n - A)$ can be expressed in the form

$$\mathrm{adj}(\lambda I_n - A) \quad = \quad \lambda^{n-1}B_{n-1} + \lambda^{n-2}B_{n-2} + \dots + \lambda B_1 + B_0,$$

for some matrices $B_0, B_1, \dots, B_{n-1} \in \mathbb{F}^{n\times n}$. Consequently,

$$\underbrace{(\lambda I_n - A)\underbrace{(\lambda^{n-1}B_{n-1} + \lambda^{n-2}B_{n-2} + \dots + \lambda B_1 + B_0)}_{=\mathrm{adj}(\lambda I_n - A)}}_{:=\mathrm{LHS}} \quad = \quad \underbrace{\det(\lambda I_n - A)I_n}_{:=\mathrm{RHS}}.$$

For the left-hand-side, we have

$$\begin{aligned}\mathrm{LHS} &= (\lambda I_n - A)(\lambda^{n-1}B_{n-1} + \dots + \lambda B_1 + B_0) \\ &= \lambda^n B_{n-1} + \lambda^{n-1}(B_{n-2} - AB_{n-1}) + \lambda^{n-2}(B_{n-3} - AB_{n-2}) + \\ &\quad + \dots + \lambda(B_0 - AB_1) - AB_0.\end{aligned}$$

For the right-hand-side, we have

$$\begin{aligned}\mathrm{RHS} &= \det(\lambda I_n - A)I_n \\ &= (\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-1} + \dots + a_1\lambda + a_0)I_n \\ &= \lambda^n I_n + \lambda^{n-1}a_{n-1}I_n + \lambda^{n-2}a_{n-2}I_n + \dots + \lambda a_1 I_n + a_0 I_n.\end{aligned}$$

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[2] By Proposition 2.4 of Lecture Notes 15, the determinant of a square matrix with two identical rows is zero.

[3] Indeed, suppose that $n = 1$, and consider any matrix $A = [a_{1,1}]$ in $\mathbb{F}^{1\times 1}$. Then $p_A(\lambda) = \det(\lambda I_1 - A) = \det(\begin{bmatrix} \lambda - a_{1,1} \end{bmatrix}) = \lambda - a_{1,1}$, and we see that $A - a_{1,1}I_1 = O_{1\times 1}$.

The corresponding coefficients in front of λ^i (for $i \in \{0, 1, \dots, n\}$) must be equal on the left-hand-side (LHS) and the right-hand-side (RHS). This yields the following $n+1$ equations.

$$\begin{array}{rcl} B_{n-1} & = & I_n \\ B_{n-2} - AB_{n-1} & = & a_{n-1}I_n \\ B_{n-3} - AB_{n-2} & = & a_{n-2}I_n \\ & \vdots & \\ B_0 - AB_1 & = & a_1 I_n \\ -AB_0 & = & a_0 I_n \end{array}$$

We now multiply the first (top) equation by A^n on the left, the second equation by A^{n-1} on the left, the third equation by A^{n-2} on the left, and so on. (The $(n+1)$-th equation, i.e. the bottom one, gets multiplied by $A^0 = I_n$ on the left, i.e. it remains unchanged). This yields the following.

$$\begin{array}{rcl} A^n B_{n-1} & = & A^n \\ A^{n-1}B_{n-2} - A^n B_{n-1} & = & a_{n-1}A^{n-1} \\ A^{n-2}B_{n-3} - A^{n-1}B_{n-2} & = & a_{n-2}A^{n-2} \\ & \vdots & \\ AB_0 - A^2 B_1 & = & a_1 A \\ -AB_0 & = & a_0 I_n \end{array}$$

We now add up the equations that we obtained. On the left-hand-side, the sum is obviously $O_{n\times n}$. So, the right-hand-side must also sum up to $O_{n\times n}$, i.e.

$$A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \dots + a_1 A + a_0 I_n \quad = \quad O_{n\times n}.$$

But this is precisely what we needed to show. ☐

Corollary 1.2. *For all matrices $A \in \mathbb{F}^{n\times n}$, both the following hold:*

(a) $A^n \in Span(I_n, A, A^2, \dots, A^{n-1})$, i.e. A^n is a linear combination of the matrices $I_n, A, A^2, \dots, A^{n-1}$;

(b) if A is invertible, then $A^{-1} \in Span(I_n, A, A^2, \dots, A^{n-1})$, i.e. A^{-1} is a linear combination of the matrices $I_n, A, A^2, \dots, A^{n-1}$.

Proof. Fix a matrix $A \in \mathbb{F}^{n\times n}$, and consider its characteristic polynomial $p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \dots + a_1\lambda + a_0$.

(a) By the Cayley-Hamilton theorem, we have that

$$A^n + a_{n-1}A^{n-1} + \dots + a_a A^2 + a_1 A + a_0 I_n \quad = \quad O_{n\times n}.$$

Consequently,

$$A^n \quad = \quad -a_0 I_n - a_1 A - a_2 A^2 - \dots - a_{n-1}A^{n-1}.$$

Thus, A^n is a linear combination of the matrices $I_n, A, A^2, \dots, A^{n-1}$, i.e. $A^n \in \text{Span}(I_n, A, A^2, \dots, A^{n-1})$.

(b) Since A is invertible, we know that 0 is not an eigenvalue of A (by Problem 3 of HW#5). Since the eigenvalues of A are precisely the roots of the characteristic polynomial of A, we have that $p_A(0) \neq 0$; since $p_A(0) = a_0$, it follows that $a_0 \neq 0$.[4]

Now, by the Cayley-Hamilton theorem, we have that

$$A^n + a_{n-1}A^{n-1} + \dots + a_2A^2 + a_1A + a_0I_n \quad = \quad O_{n\times n}.$$

We multiply both sides of the equation by A^{-1} on the right, and we obtain

$$A^{n-1} + a_{n-1}A^{n-2} + \dots + a_2A + a_1I_n + a_0A^{-1} \quad = \quad O_{n\times n},$$

and consequently,

$$a_0A^{-1} \quad = \quad -a_1I_n - a_2A - \dots - a_{n-1}A^{n-2} - A^{n-1}.$$

Since $a_0 \neq 0$, this implies that

$$A^{-1} \quad = \quad -\tfrac{a_1}{a_0}I_n - \tfrac{a_2}{a_0}A - \dots - \tfrac{a_{n-1}}{a_0}A^{n-2} - \tfrac{1}{a_0}A^{n-1}.$$

So, A^{-1} is a linear combination of the matrices $I_n, A, A^2, \dots, A^{n-1}$, i.e. $A^{n-1} \in \text{Span}(I_n, A, A^2, \dots, A^{n-1})$. ☐

2 Eigenvalues and eigenvectors of similar matrices

Matrices $A, B \in \mathbb{F}^{n\times n}$ are *similar* if there exists an invertible matrix $P \in \mathbb{F}^{n\times n}$ such that $B = P^{-1}AP$. Obviously, if $B = P^{-1}AP$, then $A = PBP^{-1} = (P^{-1})^{-1}AP^{-1}$, and so matrix similarity is a symmetric relation on $\mathbb{F}^{n\times n}$. It is also easy to check that it is reflexive and transitive,[5] and it follows that matrix similarity is an equivalence relation on $\mathbb{F}^{n\times n}$. We remark that by Theorem 4.3 of Lecture Notes 10, two matrices in $\mathbb{F}^{n\times n}$ are similar if and only if they represent the same linear transformation from an n-dimensional vector space V over $\mathbb{F}$ to itself, but possibly with respect to different bases of V.

Proposition 2.1. *Let A and B be similar matrices in $\mathbb{F}^{n\times n}$. Then A and B have the same characteristic polynomial, as well as the same eigenvalues, with the same corresponding algebraic multiplicities, and with the same corresponding geometric multiplicities.*

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[4] Indeed, 0 is not a root of $p_A(\lambda)$, and so $p_A(0) \neq 0$. But $p_A(0) = a_0$, and it follows that $a_0 \neq 0$.

[5] Indeed, any matrix $A \in \mathbb{F}^{n\times n}$ is similar to itself because $A = I_n^{-1}AI_n$, and so matrix similarity is reflexive. For transitivity, suppose that $A, B, C \in \mathbb{F}^{n\times n}$ are matrices such that A is similar to B, and B is similar to C. Then there exist invertible matrices $P, Q \in \mathbb{F}^{n\times n}$ such that $B = P^{-1}AP$ and $C = Q^{-1}BQ$. But then $C = Q^{-1}BQ = Q^{-1}P^{-1}APQ = (PQ)^{-1}A(PQ)$, and so A and C are similar.

Proof. Since A and B are similar, there exists an invertible matrix $P \in \mathbb{F}^{n\times n}$ such that $B = P^{-1}AP$. To see that A and B have the same characteristic polynomial, we compute:

$$\begin{aligned}
p_B(\lambda) &= \det(\lambda I_n - B) \\
&= \det(\lambda I_n - P^{-1}AP) \\
&= \det(P^{-1}(\lambda I_n - A)P) \\
&\overset{(*)}{=} \underbrace{\det(P^{-1})}_{\overset{(**)}{=}\frac{1}{\det(P)}} \underbrace{\det(\lambda I_n - A)}_{=p_A(\lambda)} \det(P) \\
&= p_A(\lambda),
\end{aligned}$$

where (*) follows from Theorem 1.3 of Lecture Notes 16, and (**) follows from Corollary 1.4 of Lecture Notes 16. So, A and B have the same characteristic polynomial, and consequently, the same eigenvalues with the same corresponding algebraic multiplicities.[6]

Now, fix an eigenvalue λ of A and B, and let m_A and m_B be the geometric multiplicities of λ as an eigenvalue of A and B, respectively. We must show that $m_A = m_B$. Since matrix similarity is symmetric, it suffices to show that $m_A \leq m_B$. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_{m_A}\}$ be a basis for the eigenspace $E_\lambda(A)$. Then for all $i \in \{1, \dots, m_A\}$, $P^{-1}\mathbf{v}_i$ is an eigenvector of B associated with λ, since

$$\begin{aligned}
B(P^{-1}\mathbf{v}_i) &= (\underbrace{P^{-1}AP}_{=B})(P^{-1}\mathbf{v}_i) \\
&= P^{-1}A(\underbrace{PP^{-1}}_{=I_n})\mathbf{v}_i \\
&= P^{-1}(A\mathbf{v}_i) \\
&= P^{-1}(\lambda\mathbf{v}_i) \qquad \text{because } \mathbf{v}_i \in E_\lambda(A) \\
&= \lambda(P^{-1}\mathbf{v}_i).
\end{aligned}$$

Let us now show that the set $\{P^{-1}\mathbf{v}_1, \dots, P^{-1}\mathbf{v}_{m_A}\}$ is linearly independent. Fix scalars $\alpha_1, \dots, \alpha_{m_A} \in \mathbb{F}$ such that $\alpha_1 P^{-1}\mathbf{v}_1 + \dots + \alpha_{m_A} P^{-1}\mathbf{v}_{m_A} = \mathbf{0}$. By multiplying both sides on the left by P, we obtain $\alpha_1\mathbf{v}_1 + \dots + \alpha_{m_A}\mathbf{v}_{m_A} = \mathbf{0}$.

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[6] We are using the fact that, by Theorem 2.4 of Lecture Notes 18, the eigenvalues of a square matrix are precisely the roots of the characteristic polynomial of that matrix. Moreover, by definition, the algebraic multiplicity of an eigenvalue is determined solely by the characteristic polynomial.

Since $\{\mathbf{v}_1, \dots, \mathbf{v}_{m_A}\}$ is a basis of $E_\lambda(A)$, and is therefore linearly independent, we see that $\alpha_1 = \cdots = \alpha_{m_A} = 0$. So, $\{P^{-1}\mathbf{v}_1, \dots, P^{-1}\mathbf{v}_{m_A}\}$ is linearly independent. Since $\{P^{-1}\mathbf{v}_1, \dots, P^{-1}\mathbf{v}_{m_A}\} \subseteq E_\lambda(B)$, it follows that $m_A \leq \dim(E_\lambda(B)) = m_B$, and we are done. □

Proposition 2.2. *Let $B, C \in \mathbb{F}^{n\times n}$ be similar matrices, with $C = P^{-1}BP$ for some invertible matrix $P \in \mathbb{F}^{n\times n}$. Then for all non-negative integers m, we have that*

$$C^m \quad = \quad P^{-1}B^mP \qquad \textit{and} \qquad B^m \quad = \quad PC^mP^{-1}.$$

Proof. Obviously, each of the two equalities implies the other, and so it is enough to prove one of them. Let us prove the first one. We proceed by induction on m. First, we have that $C^0 = I_n$ and that $P^{-1}B^0P = P^{-1}I_nP = I_n$, and so $C^0 = P^{-1}B^0P$. Thus, the claim holds for $m = 0$. Now, fix a non-negative integer m, and assume inductively that $C^m = P^{-1}B^mP$.Obviously, each of the two equalities implies the other, and so it is enough to prove one of them. Let us prove the first one. We proceed by induction on m. First, we have that $C^0 = I_n$ and that $P^{-1}B^0P = P^{-1}I_nP = I_n$, and so $C^0 = P^{-1}B^0P$. Thus, the claim holds for $m = 0$. Now, fix a non-negative integer m, and assume inductively that $C^m = P^{-1}B^mP$. Then

$$\begin{aligned}
C^{m+1} &= C^mC \\
&= (\underbrace{P^{-1}B^mP}_{=C^m})(\underbrace{P^{-1}BP}_{=C}) && \text{by the induction hypothesis} \\
&= P^{-1}B^m(\underbrace{PP^{-1}}_{=I_n})BP \\
&= P^{-1}B^{m+1}P.
\end{aligned}$$

This completes the induction. □

3 Algebraic and geometric multiplicities revisited

In this section, we use properties of similar matrices, as well as matrices of linear transformations with respect to convenient bases, to prove Theorem 2.5 of Lecture Notes 18.

Theorem 2.5 of Lecture Notes 18. *Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$. Then the geometric multiplicity of any eigenvalue of A is no greater than the algebraic multiplicity of that eigenvalue.*

Proof. Suppose that λ_0 is an eigenvalue of A of geometric multiplicity k. We must show that the eigenvalue λ_0 has algebraic multiplicity at least k. In view of Proposition 2.1, it is enough to exhibit a matrix $B \in \mathbb{F}^{n\times n}$ similar to A, and such that λ_0 is an eigenvalue of B of algebraic multiplicity at least k.

Since the geometric multiplicity of the eigenvalue λ_0 of A is k, we see that the eigenspace E_{λ_0} has a k-element basis, say $\{\mathbf{p}_1, \dots, \mathbf{p}_k\}$.[7] We now extend $\{\mathbf{p}_1, \dots, \mathbf{p}_k\}$ to a basis $\mathcal{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_k, \mathbf{p}_{k+1}, \dots, \mathbf{p}_n\}$ of $\mathbb{F}^n$.[8] Let $f_A : \mathbb{F}^n \to \mathbb{F}^n$ be the linear transformation whose standard matrix is A (i.e. $f_A(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{F}^n$), so that $A = {}_{\mathcal{E}_n}[f_A]_{\mathcal{E}_n}$, where $\mathcal{E}_n = \{\mathbf{e}_1^n, \dots, \mathbf{e}_n^n\}$ is the standard basis of $\mathbb{F}^n$. Consider the matrix $B := {}_{\mathcal{P}}[f_A]_{\mathcal{P}}$. By Theorem 4.3 from Lecture Notes 10, A and B are similar.[9] It now remains to show that λ_0 is an eigenvalue of B of algebraic multiplicity at least k. For this, we observe that

$$\begin{aligned}
B &= {}_{\mathcal{P}}[f_A]_{\mathcal{P}} \\
&\overset{(*)}{=} \begin{bmatrix} [f_A(\mathbf{p}_1)]_{\mathcal{P}} & \dots & [f_A(\mathbf{p}_k)]_{\mathcal{P}} & [f_A(\mathbf{p}_{k+1})]_{\mathcal{P}} & \dots & [f_A(\mathbf{p}_n)]_{\mathcal{P}} \end{bmatrix} \\
&= \begin{bmatrix} [A\mathbf{p}_1]_{\mathcal{P}} & \dots & [A\mathbf{p}_k]_{\mathcal{P}} & [f_A(\mathbf{p}_{k+1})]_{\mathcal{P}} & \dots & [f_A(\mathbf{p}_n)]_{\mathcal{P}} \end{bmatrix} \\
&\overset{(**)}{=} \begin{bmatrix} [\lambda_0\mathbf{p}_1]_{\mathcal{P}} & \dots & [\lambda_0\mathbf{p}_k]_{\mathcal{P}} & [f_A(\mathbf{p}_{k+1})]_{\mathcal{P}} & \dots & [f_A(\mathbf{p}_n)]_{\mathcal{P}} \end{bmatrix} \\
&= \begin{bmatrix} \lambda_0\mathbf{e}_1^n & \dots & \lambda_0\mathbf{e}_k^n & [f_A(\mathbf{p}_{k+1})]_{\mathcal{P}} & \dots & [f_A(\mathbf{p}_n)]_{\mathcal{P}} \end{bmatrix} \\
&= \left[\begin{array}{c|ccc} \begin{array}{c} \lambda_0 I_k \\ \hline O_{(n-k)\times k} \end{array} & [f_A(\mathbf{p}_{k+1})]_{\mathcal{P}} & \dots & [f_A(\mathbf{p}_n)]_{\mathcal{P}} \end{array}\right],
\end{aligned}$$

where (*) follows from Theorem 2.1 of Lecture Notes 10, and (**) follows from the fact that $\mathbf{p}_1, \dots, \mathbf{p}_k \in E_{\lambda_0}$. We now have that

$$p_B(\lambda) \quad = \quad \det(\lambda I_n - B) \quad = \quad \left|\begin{array}{c|c} \begin{array}{c} (\lambda-\lambda_0) I_k \\ \hline O_{(n-k)\times k} \end{array} & C \end{array}\right|,$$

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[7] Obviously, $\mathbf{p}_1, \dots, \mathbf{p}_k$ are all eigenvectors of A associated with the eigenvalue λ_0, and they form a linearly independent set of vectors in $\mathbb{F}^n$.

[8] This is possible by Proposition 1.10 of Lecture Notes 7.

[9] Here is a direct proof, which does not rely on Theorem 4.3 of Lecture Notes 10. We compute:

$$\begin{aligned}
A &= {}_{\mathcal{E}_n}[f_A]_{\mathcal{E}_n} \\
&= {}_{\mathcal{E}_n}[Id_{\mathbb{F}^n} \circ f_A \circ Id_{\mathbb{F}^n}]_{\mathcal{E}_n} \\
&\overset{(*)}{=} {}_{\mathcal{E}_n}[Id_{\mathbb{F}^n}]_{\mathcal{P}}\ {}_{\mathcal{P}}[f_A]_{\mathcal{P}}\ {}_{\mathcal{P}}[Id_{\mathbb{F}^n}]_{\mathcal{E}_n} \\
&\overset{(**)}{=} ({}_{\mathcal{P}}[Id_{\mathbb{F}^n}]_{\mathcal{E}_n})^{-1}\ \underbrace{{}_{\mathcal{P}}[f_A]_{\mathcal{P}}}_{=B}\ {}_{\mathcal{P}}[Id_{\mathbb{F}^n}]_{\mathcal{E}_n},
\end{aligned}$$

where (*) follows from Proposition 2.3 of Lecture Notes 10, and (**) follows from Proposition 2.4(c) of Lecture Notes 10. If we now set $P := {}_{\mathcal{P}}[Id_{\mathbb{F}^n}]_{\mathcal{E}_n}$, then we obtain $A = P^{-1}BP$, and so A and B are similar.

where

$$C \quad = \quad \left[\begin{array}{ccc} \lambda \mathbf{e}_{k+1} - [f_A(\mathbf{p}_{k+1})]_{\mathcal{P}} & \dots & \lambda \mathbf{e}_n - [f_A(\mathbf{p}_{k+1})]_{\mathcal{P}} \end{array} \right]_{(n-k)\times n}.$$

Thus, $p_B(\lambda)$ has the form

$$p_B(\lambda) \quad = \quad \left| \begin{array}{cccc:cccc} \lambda-\lambda_0 & 0 & \dots & 0 & * & * & \dots & * \\ 0 & \lambda-\lambda_0 & \dots & 0 & * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda-\lambda_0 & * & * & \dots & * \\ \hdashline 0 & 0 & \dots & 0 & * & * & \dots & * \\ 0 & 0 & \dots & 0 & * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & * & * & \dots & * \end{array} \right|,$$

where the red matrix in the upper-left corner (to the left of the vertical dotted line, and above the horizontal dotted line) is of size $k \times k$. By iteratively performing Laplace expansion along the first column, we see that $p_B(\lambda)$ has a factor $(\lambda - \lambda_0)^k$, and consequently, λ_0 is an eigenvalue of B of algebraic multiplicity at least k. This completes the argument. ☐

4 Diagonal matrices and diagonalization

A matrix $A = \left[\begin{array}{c} a_{i,j} \end{array} \right]_{n\times n}$ in $\mathbb{F}^{n\times n}$ is *diagonal* if $a_{i,j} = 0$ for all distinct $i, j \in \{1, \dots, n\}$. In other words, a square matrix is diagonal if all its entries off the main diagonal are zero (and the main diagonal is arbitrary).

Given scalars $\lambda_1, \dots, \lambda_n \in \mathbb{F}$, we will denote by $D(\lambda_1, \dots, \lambda_n)$ the diagonal matrix in which $\lambda_1, \dots, \lambda_n$ appear on the main diagonal (in that order), i.e.

$$\begin{aligned} D(\lambda_1, \lambda_2, \dots, \lambda_n) \quad &:= \quad \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \\ &= \quad \left[\begin{array}{cccc} \lambda_1 \mathbf{e}_1 & \lambda_2 \mathbf{e}_2 & \dots & \lambda_n \mathbf{e}_n \end{array} \right]. \end{aligned}$$

Proposition 4.1. *Let $\lambda_1, \dots, \lambda_n \in \mathbb{F}^n$, and set $D := D(\lambda_1, \dots, \lambda_n)$. Then both the following hold:*

(a) For all matrices $P = \left[\begin{array}{cc} \mathbf{p}_1 & \dots \mathbf{p}_n \end{array} \right]$ in $\mathbb{F}^{n\times n}$, we have that

$$PD \quad = \quad \left[\begin{array}{cc} \lambda_1 \mathbf{p}_1 & \dots \lambda_n \mathbf{p}_n \end{array} \right];$$

(b) For all non-negative integers m, we have that

$$D^m = D(\lambda_1^m, \dots, \lambda_n^m).$$

Proof. (a) Fix a matrix $P = \begin{bmatrix} \mathbf{p}_1 & \dots & \mathbf{p}_n \end{bmatrix}$ in $\mathbb{F}^{n \times n}$. Then

$$\begin{aligned} PD &= P \begin{bmatrix} \lambda_1 \mathbf{e}_1 & \dots & \lambda_n \mathbf{e}_n \end{bmatrix} \\ &= \begin{bmatrix} P(\lambda_1 \mathbf{e}_1) & \dots & P(\lambda_n \mathbf{e}_n) \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 (P\mathbf{e}_1) & \dots & \lambda_n (P\mathbf{e}_n) \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 \mathbf{p}_1 & \dots & \lambda_n \mathbf{p}_n \end{bmatrix}. \end{aligned}$$

(b) This readily follows from (a) via an easy induction on m. Indeed, $D^0 = I_n = D(\lambda_1^0, \dots, \lambda_n^0)$, and so the statement is true for $m = 0$. Now, fix a non-negative integer m, and assume inductively that $D^m = D(\lambda_1^m, \dots, \lambda_n^m)$. We now compute:

$$\begin{aligned} D^{m+1} &= \underbrace{D(\lambda_1^m, \dots, \lambda_n^m)}_{=D^m} \underbrace{\begin{bmatrix} \lambda_1 \mathbf{e}_1 & \dots & \lambda_n \mathbf{e}_n \end{bmatrix}}_{=D} && \text{by the induction hypothesis} \\ &= \begin{bmatrix} \lambda_1^m (\lambda_1 \mathbf{e}_1) & \dots & \lambda_n^m (\lambda_n \mathbf{e}_n) \end{bmatrix} && \text{by (a)} \\ &= \begin{bmatrix} \lambda_1^{m+1} \mathbf{e}_1 & \dots & \lambda_n^{m+1} \mathbf{e}_n \end{bmatrix} \\ &= D(\lambda_1^{m+1}, \dots, \lambda_n^{m+1}). \end{aligned}$$

This completes the induction. □

4.1 Diagonalizable matrices

A matrix $A \in \mathbb{F}^{n \times n}$ is *diagonalizable* if it is similar to some diagonal matrix in $\mathbb{F}^{n \times n}$. To *diagonalize* a matrix $A \in \mathbb{F}^{n \times n}$ means to find a diagonal matrix D and an invertible matrix P, both in $\mathbb{F}^{n \times n}$, such that $D = P^{-1}AP$.

Theorem 4.2. *A matrix $A \in \mathbb{F}^{n \times n}$ is diagonalizable if and only if $\mathbb{F}^n$ has a basis formed by eigenvectors of A.*

Proof. Fix a matrix $A \in \mathbb{F}^{n \times n}$. Suppose first that A is diagonalizable, and fix matrices $P, D \in \mathbb{F}^{n \times n}$ such that D is diagonal, P is invertible, and $D = P^{-1}AP$. Since P is invertible, its columns form a basis of $\mathbb{F}^{n \times n}$.[10] So, it is enough to show that every column of P is an eigenvector of A. Set

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[10] This follows from Theorem 4.1 of Lecture Notes 7.

$P = \begin{bmatrix} \mathbf{p}_1 & \dots & \mathbf{p}_n \end{bmatrix}$ and $D = D(\lambda_1, \dots, \lambda_n)$. Since $D = P^{-1}AP$, we have that $PD = AP$. For each $i \in \{1, \dots, n\}$, the i-th column of PD is $\lambda_i \mathbf{p}_i$ (by Proposition 4.1(a)), and the i-th column of AP is $A\mathbf{p}_i$, and consequently, $A\mathbf{p}_i = \lambda_i \mathbf{p}_i$, i.e. λ_i is an eigenvalue of A, and $\mathbf{p}_i$ is an eigenvector of A associated with λ_i.[11] So, the columns of P are eigenvectors of A.

Suppose now that $\mathbb{F}^n$ has a basis, say, $\mathcal{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$, formed by eigenvectors of A. We must show that A is diagonalizable. For each $i \in \{1, \dots, n\}$, let λ_i be the eigenvalue of A associated with the eigenvector $\mathbf{p}_i$, so that $A\mathbf{p}_i = \lambda_i \mathbf{p}_i$. Set $D = D(\lambda_1, \dots, \lambda_n)$ and $P = \begin{bmatrix} \mathbf{p}_1 & \dots & \mathbf{p}_n \end{bmatrix}$. The matrix D is diagonal (by construction), and the matrix P is invertible because its columns form a basis of $\mathbb{F}^n$.[12] We will show that $D = P^{-1}AP$. Here, we give two proofs. The first one is more elementary, but the second one gives a better geometric intuition.

Proof #1. Note that for each $i \in \{1, \dots, n\}$, the i-th column of AP is $A\mathbf{p}_i$, the i-th column of PD is $\lambda_i \mathbf{p}_i$ (by Proposition 4.1(a)), and we know that $A\mathbf{p}_i = \lambda_i \mathbf{p}_i$. So, AP and PD have the same corresponding columns, and consequently, $AP = PD$. We multiply both sides by P^{-1} on the left, and we obtain $P^{-1}AP = D$; since D is diagonal, it follows that A is diagonalizable.

Proof #2. Consider the unique linear transformation $f : \mathbb{F}^n \to \mathbb{F}^n$ such that $f(\mathbf{p}_1) = \lambda_1 \mathbf{p}_1, \dots, f(\mathbf{p}_n) = \lambda_n \mathbf{p}_n$.[13] Now, consider the linear transformation $f_A : \mathbb{F}^n \to \mathbb{F}^n$ given by $f_A(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbf{v}$ (so, A is the standard matrix of f_A). Then $f_A(\mathbf{p}_1) = A\mathbf{p}_1 = \lambda_1 \mathbf{p}_1, \dots, f_A(\mathbf{p}_1) = A\mathbf{p}_n = \lambda_n \mathbf{p}_n$. The uniqueness of f guarantees that $f = f_A$, and consequently, that

$$_{\mathcal{E}_n}[f]_{\mathcal{E}_n} \quad = \quad _{\mathcal{E}_n}[f_A]_{\mathcal{E}_n} \quad = \quad A,$$

where $\mathcal{E}_n$ is the standard basis of $\mathbb{F}^n$. On the other hand, we have that

$$\begin{aligned} _{\mathcal{P}}[f]_{\mathcal{P}} &\overset{(*)}{=} \begin{bmatrix} [f(\mathbf{p}_1)]_{\mathcal{P}} & \dots & [f(\mathbf{p}_n)]_{\mathcal{P}} \end{bmatrix} \\ &= \begin{bmatrix} [\lambda_1 \mathbf{p}_1]_{\mathcal{P}} & \dots & [\lambda_n \mathbf{p}_n]_{\mathcal{P}} \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 \mathbf{e}_1 & \dots & \lambda_n \mathbf{e}_n \end{bmatrix} \\ &= D(\lambda_1, \dots, \lambda_n) \\ &= D, \end{aligned}$$

where (*) follows from Theorem 2.1 of Lecture Notes 10. We now compute:

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[11] We are also using the fact that $\mathbf{p}_i \neq \mathbf{0}$. But this follows immediately from the fact that $\{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ is a basis of $\mathbb{F}^n$.

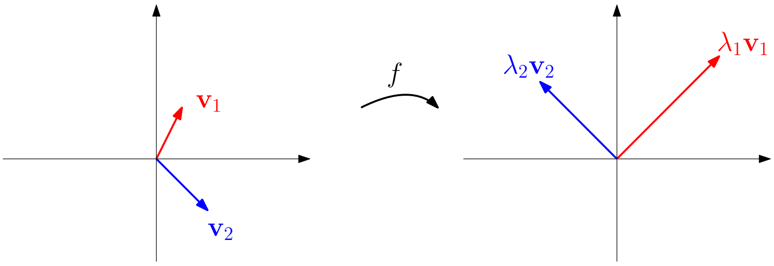
[12] Once again, this follows from Theorem 4.1 of Lecture Notes 7.

[13] The existence and uniqueness of f are guaranteed by Theorem 2.2 of Lecture Notes 9.

$$\begin{aligned} D &= {}_{\mathcal{P}}[f]_{\mathcal{P}} \\ &= {}_{\mathcal{P}}[\mathrm{Id}_{\mathbb{F}^n} \circ f \circ \mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{P}} \\ &\overset{(*)}{=} {}_{\mathcal{P}}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{E}_n} \ {}_{\mathcal{E}_n}[f]_{\mathcal{E}_n} \ {}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{P}} \\ &\overset{(**)}{=} P^{-1}AP, \end{aligned}$$

where (*) follows from Proposition 2.3 of Lecture Notes 10, and where in (**), we used the fact that ${}_{\mathcal{E}_n}[f]_{\mathcal{E}_n} = A$ (proven above), and the fact that ${}_{\mathcal{P}}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{E}_n} = P^{-1}$ and ${}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{P}} = P$ (which follows from Lemma 2.6 of Lecture Notes 10).

Remark: Proof #2 shows that A is the standard matrix of the unique linear transformation $f : \mathbb{F}^n \to \mathbb{F}^n$ defined by scaling each of the vectors in the basis $\mathcal{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ by a suitable scalar (eigenvalue), and the diagonal matrix D is the matrix of this linear transformation with respect to the basis $\mathcal{P}$ itself. As shown above, A and D are similar.[14] For the case when $n = 2$, this is illustrated below.



□

In view of Theorem 4.2, it may be useful to recall when $\mathbb{F}^n$ has a basis formed by eigenvectors of $A^{n\times n}$. This is precisely described in Corollary 2.13 of Lecture Notes 18, restated below.

Corollary 2.13 of Lecture Notes 18. *Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$. Then the following are equivalent:*

(i) $\mathbb{F}^n$ has a basis formed by eigenvectors of A;

(ii) the sum of algebraic multiplicities of all distinct eigenvalues A is equal to n, and the geometric multiplicity of each eigenvalue of A is equal to the algebraic multiplicity of that eigenvalue;

(iii) the sum of geometric multiplicities of all distinct eigenvalues A is equal to n.

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[14] The similarity of A and D also follows from Theorem 4.3 of Lecture Notes 10.

Theorem 4.2 (of the present lecture notes) and Corollary 2.13 of Lecture Notes 18 immediately yield the following corollary.

Corollary 4.3. *Let* $A \in \mathbb{F}^{n \times n}$ *be a matrix that has* n *distinct eigenvalues. Then* A *is diagonalizable.*

Proof. Since A has n distinct eigenvalues, and each of those eigenvalues has geometric multiplicity at least 1 (by the definition of an eigenvalue), we see that the sum of geometric multiplicities of the distinct eigenvalues of A is at least n. But then the sum of geometric multiplicities of the distinct eigenvalues of A must be exactly n.[15] Now Corollary 2.13 of Lecture Notes 18 guarantees that $\mathbb{F}^n$ has a basis formed by eigenvectors of A, and so by Theorem 4.2, A is diagonalizable. $\square$

The theory we have developed so far (including both the statement and the proof of Theorem 4.2) gives us a recipe for determining whether a matrix $A \in \mathbb{F}^{n \times n}$ is diagonalizable, and if so, diagonalizing it. We proceed as follows (the steps are in numerals, and justifications of individual steps are given in bullet points underneath the steps in question).

1. We compute the characteristic polynomial $p_A(\lambda)$, factor it, and compute all the eigenvalues of A together with their algebraic multiplicities.

2. If the algebraic multiplicities of the eigenvalues of A add up to less than n, then A is not diagonalizable.

    - **Note:** The sum of algebraic multiplicities of the eigenvalues of A will always be at most n, and if the field $\mathbb{F}$ is algebraically closed (for example, if $\mathbb{F} = \mathbb{C}$), then it will be equal to n. If $\mathbb{F}$ is **not** algebraically closed (for example, if $\mathbb{F}$ is $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{Z}_p$ for some prime number p), then it is possible that the sum of algebraic multiplicities of the eigenvalues of A is less than n. In this case, Corollary 2.13 of Lecture Notes 18 guarantees that $\mathbb{F}$ does not have a basis formed by eigenvectors of A, and consequently (by Theorem 4.2), A is not diagonalizable.

3. From now on, assume that the algebraic multiplicities of the eigenvalues of A add up to n. For each eigenvalue λ of A, find a basis of the eigenspace E_λ and the geometric multiplicity of λ.

4. If the geometric multiplicity of some eigenvalue of A is less than its algebraic multiplicity, then A is not diagonalizable.

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[15] Indeed, the sum of geometric multiplicities of the distinct eigenvalues is at most n. This is because the sum of algebraic multiplicities of the eigenvalues of A is at most n, and by Theorem 2.5 fof Lecture Notes 18, the geometric multiplicity of any eigenvalue is no greater than the algebraic multiplicity of that eigenvalue.

- **Note:** If the geometric multiplicity of some eigenvalue of A is less than its algebraic multiplicity, then Corollary 2.13 of Lecture Notes 18 guarantees that $\mathbb{F}^n$ has no basis formed by eigenvectors of A, and so by Theorem 4.2, A is not diagonalizable.

5. From now on, assume that the geometric multiplicity of each eigenvalue of A is equal to its algebraic multiplicity, so that the geometric multiplicities of the eigenvalues of A add up to n.[16] In this case, A is diagonalizable.

   - **Note:** Indeed, if the geometric multiplicities of the eigenvectors of A add up to n, then Corollary 2.13 of Lecture Notes 18 guarantees that $\mathbb{F}^n$ has a basis formed by eigenvectors of A, and so by Theorem 4.2, A is diagonalizable.

6. Let $\lambda_1, \dots, \lambda_\ell$ be the distinct eigenvalues of A, with geometric multiplicities $n_1, \dots, n_\ell$, respectively. (So, $n_1 + \dots + n_\ell = n$.) We now form the diagonal matrix $D = D(\underbrace{\lambda_1, \dots, \lambda_1}_{n_1}, \dots, \underbrace{\lambda_\ell, \dots, \lambda_\ell}_{n_\ell})$, and we form the matrix P as follows: the first n_1 columns of P form a basis of the eigenspace E_{λ_1}, the next n_2 columns of P form a basis of the eigenspace E_{λ_2}, and so on, until the last n_ℓ columns of P form a basis of the eigenspace E_{λ_ℓ}. Now $D = PAP^{-1}$

   - **Note:** Theorem 2.12 of Lecture Notes 18 guarantees that the columns of P form a linearly independent set; since P has n columns, it follows that the columns of P form a basis of $\mathbb{F}^n$. Now the correctness of our construction follows from the proof of Theorem 4.2.

Example 4.4. *Consider the matrix*

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

in $\mathbb{C}^{3\times 3}$. Determine whether A is diagonalizable, and if so, diagonalize it.

Solution. The matrix A is precisely the matrix from Example 2.6 of Lecture Notes 18. In that example, we determined that A has two eigenvalues, namely, $\lambda_1 = 4$ (with algebraic multiplicity 1 and geometric multiplicity 1) and $\lambda_2 = 5$ (with algebraic multiplicity 2 and geometric multiplicity 2). Since the sum of geometric multiplicities is 3, we see that the 3×3 matrix

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[16] We already assumed that the algebraic multiplicities of the eigenvalues of A add up to n. So, if the geometric multiplicity of each eigenvalue of A is equal to the algebraic multiplicity of that eigenvalue, then the geometric multiplicities of the eigenvalues of A add up to n.

A is indeed diagonalizable. In Example 2.6 of Lecture Notes 18, we saw that $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ is a basis of the eigespace E_{λ_1}, and that $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of the eigenspace E_{λ_2}. So, we set

$$D \;:=\; \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad \text{and} \qquad P \;:=\; \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and we see that $D = P^{-1}AP$. □

Example 4.5. *Consider the matrix*

$$A \;=\; \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

in $\mathbb{C}^{5\times 5}$. *Determine whether* A *is diagonalizable, and if so, diagonalize it.*

Solution. The matrix A is precisely the matrix from Example 2.10 of Lecture Notes 18. In that example, we determined that A has three eigenvalues, namely $\lambda_1 = 1$ (with algebraic multiplicity 2 and geometric multiplicity 2), $\lambda_2 = 2$ (with algebraic multiplicity 1 and geometric multiplicity 1), and $\lambda_3 = 3$ (with algebraic multiplicity 2 and geometric multiplicity 1). Since the geometric multiplicity of the eigenvalue $\lambda_3 = 3$ is strictly smaller than the algebraic multiplicity, we see that A is not diagonalizable. □

4.2 Exponentiating diagonalizable matrices

One reason we care about diagonalizability is because diagonalizable matrices are easy to exponentiate. Indeed, suppose we are given a diagonalizable matrix $A \in \mathbb{F}^{n\times n}$. Suppose, furthermore, that we have diagonalized A as $D = P^{-1}AP$, where $D = D(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix and P an invertible matrix in $\mathbb{F}^{n\times n}$. Then by Proposition 2.2, for all non-negative integers m, we have that $A^m = PD^mP^{-1}$. On the other hand, by Proposition 4.1(b), we have that $D^m = D(\lambda_1^m, \ldots, \lambda_n^m)$. So, A^m is easy to compute.

Example 4.6. *Consider the matrix*

$$A \;=\; \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

in $\mathbb{C}^{3\times 3}$. *Find a formula for* A^m *(where* m *is an arbitrary non-negative integer).*

Solution. This is the matrix from Example 4.4. In that example, we computed matrices $D, P \in \mathbb{C}^{3\times 3}$ such that D is diagonal, P is invertible, and $D = P^{-1}AP$. The matrices in question were

$$D \quad := \quad \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad \text{and} \qquad P \quad := \quad \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We then compute

$$P^{-1} \quad = \quad \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then for all non-negative integers m, we have the following:

$$\begin{aligned} A^m \quad &\overset{(*)}{=} \quad PD^mP^{-1} \\ &\overset{(**)}{=} \quad \underbrace{\begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=P} \underbrace{\begin{bmatrix} 4^m & 0 & 0 \\ 0 & 5^m & 0 \\ 0 & 0 & 5^m \end{bmatrix}}_{=D^m} \underbrace{\begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}}_{=P^{-1}} \\ &\overset{(***)}{=} \quad \begin{bmatrix} 4^m & 0 & 2\cdot 4^m - 2\cdot 5^m \\ -2\cdot 4^m + 2\cdot 5^m & 5^m & -4^{m+1} + 4\cdot 5^m \\ 0 & 0 & 5^m \end{bmatrix} \\ &= \quad \begin{bmatrix} 4^m & 0 & 2(4^m - 5^m) \\ 2(5^m - 4^m) & 5^m & 4(5^m - 4^m) \\ 0 & 0 & 5^m \end{bmatrix}, \end{aligned}$$

where (*) follows from Proposition 2.2, (**) follows from Proposition 4.1, and (***) follows by simple matrix multiplication.

Optional: We can check that our answer is correct by induction, as follows. For $m = 0$, we have

$$\begin{bmatrix} 4^0 & 0 & 2(4^0 - 5^0) \\ 2(5^0 - 4^0) & 5^0 & 4(5^0 - 4^0) \\ 0 & 0 & 5^0 \end{bmatrix} \quad = \quad I_3 \quad = \quad A^0.$$

Now, fix a non-negative integer m, and assume inductively that

$$A^m \quad = \quad \begin{bmatrix} 4^m & 0 & 2(4^m - 5^m) \\ 2(5^m - 4^m) & 5^m & 4(5^m - 4^m) \\ 0 & 0 & 5^m \end{bmatrix}.$$

We now compute:

$$\begin{aligned}
A^{m+1} &= A^m A \\
&\overset{(*)}{=} \begin{bmatrix} 4^m & 0 & 2(4^m - 5^m) \\ 2(5^m - 4^m) & 5^m & 4(5^m - 4^m) \\ 0 & 0 & 5^m \end{bmatrix} \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix} \\
&\overset{(**)}{=} \begin{bmatrix} 4^{m+1} & 0 & 8 \cdot 4^m - 10 \cdot 5^m \\ 10 \cdot 5^m - 8 \cdot 4^m & 5^{m+1} & 20 \cdot 5^m - 16 \cdot 4^m \\ 0 & 0 & 5^{m+1} \end{bmatrix} \\
&= \begin{bmatrix} 4^{m+1} & 0 & 2(4^{m+1} - 5^{m+1}) \\ 2(5^{m+1} - 4^{m+1}) & 5^{m+1} & 4(5^{m+1} - 4^{m+1}) \\ 0 & 0 & 5^{m+1} \end{bmatrix},
\end{aligned}$$

where (*) follows from the induction hypothesis, and (**) follows via simple matrix multiplication. This completes the induction and proves that our formula is correct. $\square$