Linear Algebra 2

Lecture #18
Eigenvectors and eigenvalues

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April 6, 2023

- This lecture has two parts:

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    1. algebraically closed fields;

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  2. eigenvectors and eigenvalues.

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- The Fundamental Theorem of Algebra can be restated as saying that the field $\mathbb{C}$ is algebraically closed.
- Other fields that we have studied (namely, $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}_p$ for a prime number p) are **not** algebraically closed.
- There exist algebraically closed fields other than $\mathbb{C}$, but we shall not study them in this course.

- If $\mathbb{F}$ is any algebraically closed field, and $p(x)$ is any polynomial of degree $n \geq 1$ with coefficients in $\mathbb{F}$, then there exist scalars $a, \alpha_1, \dots, \alpha_\ell \in \mathbb{F}$ s.t. $a \neq 0$ and s.t. $\alpha_1, \dots, \alpha_\ell$ are pairwise distinct, and positive integers $n_1, \dots, n_\ell$ satisfying $n_1 + \dots + n_\ell = n$, s.t.

$$p(x) = a(x - \alpha_1)^{n_1} \dots (x - \alpha_\ell)^{n_\ell}.$$

  Here, a is the leading coefficient of $p(x)$, i.e. the coefficient in front of x^n. Scalars $\alpha_1, \dots, \alpha_\ell$ are the roots of $p(x)$ with *multiplicities* $n_1, \dots, n_\ell$, respectively.

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- However, if $\mathbb{F}$ is a field that is **not** algebraically closed, then there exist non-constant polynomials $p(x)$ with coefficients in $\mathbb{F}$ that **cannot** be factored into linear factors as above.
- In our study of eigenvalues and eigenvectors, we will need to factor polynomials at various stages.
  - So, algebraically closed fields are of particular interest in the context of eigenvectors and eigenvalues.

- Recall that for a complex number $\mathbf{z} = a + ib$ (with $a, b \in \mathbb{R}$), we define the *complex conjugate* of z to be $\overline{z} = a - ib$.

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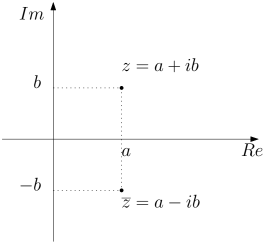

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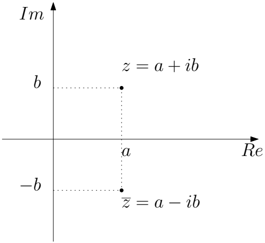

- Note that if z is a real number, then $\overline{z} = z$.

Theorem 1.1

Let $p(x)$ be any polynomial with real coefficients, and let $z \in \mathbb{C}$. Then z is a root of $p(x)$ iff $\overline{z}$ is a root of $p(x)$.

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Proof. Set $p(x) = a_n x^n + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_n \in \mathbb{R}$.

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Proof. Set $p(x) = a_n x^n + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_n \in \mathbb{R}$. Then we have the following sequence of equivalences:

$$\begin{aligned} p(z) = 0 \quad &\iff \quad \overline{p(z)} = \overline{0} \\ &\iff \quad \overline{a_n z^n + \cdots + a_1 z + a_0} = \overline{0} \\ &\overset{(*)}{\iff} \quad \overline{a_n}(\overline{z})^n + \cdots + \overline{a_1}(\overline{z}) = \overline{0} \\ &\overset{(**)}{\iff} \quad a_n(\overline{z})^n + \cdots + a_1\overline{z} + a_0 = 0 \\ &\iff \quad p(\overline{z}) = 0, \end{aligned}$$

where (*) follows the properties of the complex conjugate, and (**) follows from the fact that $a_0, a_1, \ldots, a_n$ and 0 are real numbers. Q.E.D.

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- It is not hard to show that the multiplicity of any complex root z of a polynomial $p(x)$ with real coefficients is equal to the multiplicity of the root $\overline{z}$, but we omit the details.

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- It is not hard to show that the multiplicity of any complex root z of a polynomial $p(x)$ with real coefficients is equal to the multiplicity of the root $\overline{z}$, but we omit the details.
- However, this only works if $p(x)$ has **real** coefficients! If $p(x)$ has complex coefficients, then Theorem 1.1 does not apply.

2. Eigenvectors and eigenvalues

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- By definition, $\mathbf{0}$ is **not** an eigenvector of any matrix $A \in \mathbb{F}^{n \times n}$.
    - This is because for any matrix $A \in \mathbb{F}^{n \times n}$ and any scalar λ, we have $A\mathbf{0} = \lambda\mathbf{0}$; if we allowed $\mathbf{0}$ to count as an eigenvector, then any scalar in $\mathbb{F}$ would be an eigenvalue, which would not be very interesting.

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- However, the scalar 0 may possibly be an eigenvalue of a square matrix A.

Example 2.1

(a) Consider the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ in $\mathbb{R}^{2\times 2}$. Note that A is the standard matrix of reflection about the x_1-axis in $\mathbb{R}^2$. The matrix A has two eigenvalues:

- $\lambda_1 = 1$ is an eigenvalue of A, and $\mathbf{v}_1 = \mathbf{e}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ is an associated eigenvector, since

$$A\mathbf{v}_1 \quad = \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad = \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad = \quad \lambda_1 \mathbf{v}_1;$$

- $\lambda_2 = -1$ is an eigenvalue of A, and $\mathbf{v}_2 = \mathbf{e}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ is an associated eigenvector, since

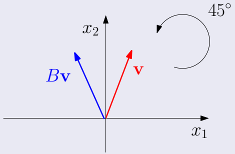
$$A\mathbf{v}_2 \quad = \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad = \quad \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad = \quad \lambda_2 \mathbf{v}_2.$$

Example 2.1

(b) Consider the matrix

$$B = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

in $\mathbb{R}^{2\times 2}$. Note that B is the standard matrix of counterclockwise rotation by 45° in $\mathbb{R}^2$.



The matrix B has no (real) eigenvalues, since it does not merely scale any non-zero vector in $\mathbb{R}^2$.

Example 2.1

(c) Consider the matrix

$$C = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

in $\mathbb{C}^{2\times 2}$. Then:

- $\lambda_1 = \frac{1+i}{\sqrt{2}}$ is an eigenvalue of C, and $\mathbf{v}_1 = \begin{bmatrix} i & 1 \end{bmatrix}^T$ is an associated eigenvector, since

$$C\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1+i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} \end{bmatrix} = \lambda_1 \mathbf{v}_1;$$

- $\lambda_2 = \frac{1-i}{\sqrt{2}}$ is an eigenvalue of C, and $\mathbf{v}_2 = \begin{bmatrix} -i & 1 \end{bmatrix}^T$ is an associated eigenvector, since

$$C\mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1-i}{\sqrt{2}} \\ \frac{1-i}{\sqrt{2}} \end{bmatrix} = \lambda_2 \mathbf{v}_2.$$

Example 2.1

(d) For any field $\mathbb{F}$ and scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{F}$, consider the diagonal matrix

$$D(\lambda_1, \lambda_2, \ldots, \lambda_n) := \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \mathbf{e}_1 & \lambda_2 \mathbf{e}_2 & \ldots & \lambda_n \mathbf{e}_n \end{bmatrix}.$$

Then $\lambda_1, \ldots, \lambda_n$ are all eigenvalues of $D(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Indeed, for each $i \in \{1, 2, \ldots, n\}$, the i-th standard vector $\mathbf{e}_i$ is an eigenvector of $D(\lambda_1, \lambda_2, \ldots, \lambda_n)$ associated with the eigenvalue λ_i, since

$$D(\lambda_1, \lambda_2, \ldots, \lambda_n)\mathbf{e}_i = \lambda_i \mathbf{e}_i.$$

- Given a field $\mathbb{F}$, a matrix $A \in \mathbb{F}^{n \times n}$, and an eigenvalue λ of A, the *eigenspace* of A associated with λ is the set

$$E_\lambda(A) \quad := \quad \{\mathbf{v} \in \mathbb{F}^n \mid A\mathbf{v} = \lambda\mathbf{v}\}.$$

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- So, the elements of $E_\lambda(A)$ are all the eigenvectors of A associated with λ, plus the zero vector.
    - We have that $\mathbf{0} \in E_\lambda(A)$ because $A\mathbf{0} = \lambda\mathbf{0}$.

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  - We have that $\mathbf{0} \in E_\lambda(A)$ because $A\mathbf{0} = \lambda\mathbf{0}$.
- When the matrix A is clear from context, we write just E_λ instead of $E_\lambda(A)$.

Theorem 2.2

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times n}$, and let λ be an eigenvalue of A. Then E_λ is a subspace of $\mathbb{F}^n$.

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Proof. 1. We have that $A\mathbf{0} = \mathbf{0} = \lambda\mathbf{0}$, and so $\mathbf{0} \in E_\lambda$.

2. Fix $\mathbf{u}, \mathbf{v} \in E_\lambda$. Then

$$\begin{aligned} A(\mathbf{u} + \mathbf{v}) &= A\mathbf{u} + A\mathbf{v} \\ &= \lambda\mathbf{u} + \lambda\mathbf{v} \qquad \text{because } \mathbf{u}, \mathbf{v} \in E_\lambda \\ &= \lambda(\mathbf{u} + \mathbf{v}), \end{aligned}$$

and so $\mathbf{u} + \mathbf{v} \in E_\lambda$.

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Proof (continued). 3. Fix $\mathbf{u} \in E_\lambda$ and $\alpha \in \mathbb{F}$. Then

$$\begin{aligned} A(\alpha\mathbf{u}) &= \alpha(A\mathbf{u}) \\ &= \alpha(\lambda\mathbf{u}) \qquad \text{because } \mathbf{u} \in E_\lambda \\ &= \lambda(\alpha\mathbf{u}), \end{aligned}$$

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and so $\alpha\mathbf{u} \in E_\lambda$.

We have now verified that E_λ satisfies the hypotheses of Theorem 2.7 of Lecture Notes 6, and so by that theorem, E_λ is indeed a subspace of $\mathbb{F}^n$.

- Given a field $\mathbb{F}$, a matrix $A \in \mathbb{F}^{n \times n}$, and an eigenvalue λ of A, the *eigenspace* of A associated with λ is the set

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- For a field $\mathbb{F}$ and a matrix $A \in \mathbb{F}^{n \times n}$, the *geometric multiplicity* of an eigenvalue λ of A is defined to be $\dim(E_\lambda)$.

Definition

- Given a field $\mathbb{F}$ and a matrix $A \in \mathbb{F}^{n\times n}$, the *characteristic polynomial* of A is defined to be

$$p_A(\lambda) \quad := \quad \det(\lambda I_n - A).$$

- The *characteristic equation* of A is the equation

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- So, the roots of the characteristic polynomial of A are precisely the solutions of the characteristic equation of A.

Example 2.3

Compute the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 0 & 2 \\ 2 & -1 & -3 \end{bmatrix}$$

in $\mathbb{C}^{3\times 3}$.

Solution.

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in $\mathbb{C}^{3\times 3}$.

Solution. The characteristic polynomial of A is:

$$\begin{aligned} p_A(\lambda) &= \det(\lambda I_3 - A) \\ &= \begin{vmatrix} \lambda - 1 & 2 & -3 \\ 1 & \lambda & -2 \\ -2 & 1 & \lambda + 3 \end{vmatrix} \\ &= \lambda^3 + 2\lambda^2 - 9\lambda - 3. \end{aligned}$$

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- In some texts, the characteristic polynomial is defined to be $\det(A - \lambda I_n)$.
- But note that $\det(\lambda I_n - A) = (-1)^n \det(A - \lambda I_n)$, and so the polynomials $\det(\lambda I_n - A)$ and $\det(A - \lambda I_n)$ have exactly the same roots, with the same corresponding multiplicities, which is what we will actually care about when it comes to the characteristic polynomial.

Definition

Given a field $\mathbb{F}$ and a matrix $A \in \mathbb{F}^{n\times n}$, the *characteristic polynomial* of A is defined to be $p_A(\lambda) := \det(\lambda I_n - A)$.

- For a field $\mathbb{F}$ and a matrix $A \in \mathbb{F}^{n\times n}$, the characteristic polynomial $p_A(\lambda) = \det(\lambda I_n - A)$ is a polynomial of degree n, with leading coefficient 1.
- In some texts, the characteristic polynomial is defined to be $\det(A - \lambda I_n)$.
- But note that $\det(\lambda I_n - A) = (-1)^n \det(A - \lambda I_n)$, and so the polynomials $\det(\lambda I_n - A)$ and $\det(A - \lambda I_n)$ have exactly the same roots, with the same corresponding multiplicities, which is what we will actually care about when it comes to the characteristic polynomial.
- The main advantage of using $\det(\lambda I_n - A)$ rather than $\det(A - \lambda I_n)$ is that the former polynomial has leading coefficient 1, whereas the latter has leading coefficient $(-1)^n$, which is -1 if n is odd.

Theorem 2.4

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times n}$, and let $\lambda_0 \in \mathbb{F}$. Then the following are equivalent:

1. (i) λ_0 is an eigenvalue of A;
2. (ii) λ_0 is a root of the characteristic polynomial of A, i.e. $p_A(\lambda_0) = 0$;
3. (iii) λ_0 is a solution of the characteristic equation of A, i.e. $\det(\lambda_0 I_n - A) = 0$.

Proof.

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Proof. The fact that (ii) and (iii) are equivalent follows immediately from the definition of the characteristic polynomial and the characteristic equation of A.

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Proof. The fact that (ii) and (iii) are equivalent follows immediately from the definition of the characteristic polynomial and the characteristic equation of A.

It remains to prove that (i) and (iii) are equivalent. For this, we have the following sequence of equivalences (next slide):

Proof (continued).

$$\begin{aligned}
\lambda_0 \text{ is an eigenvalue of } A &\iff \exists \mathbf{v} \in \mathbb{F}^n \setminus \{\mathbf{0}\} \text{ s.t. } A\mathbf{v} = \lambda_0 \mathbf{v} \\
&\iff \exists \mathbf{v} \in \mathbb{F}^n \setminus \{\mathbf{0}\} \text{ s.t. } (\lambda_0 I_n - A)\mathbf{v} = \mathbf{0} \\
&\iff \text{the equation } (\lambda_0 I_n - A)\mathbf{x} = \mathbf{0} \\
&\qquad \text{has a non-trivial solution} \\
&\overset{(*)}{\iff} \text{the matrix } \lambda_0 I_n - A \\
&\qquad \text{is non-invertible} \\
&\overset{(**)}{\iff} \det(\lambda_0 I_n - A) = 0,
\end{aligned}$$

where (*) follows from Corollary 5.1 of Lecture Notes 4, and (**) follows from Theorem 5.1 from Lecture Notes 15. This proves that (i) and (iii) are equivalent, and we are done. Q.E.D.

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Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times n}$, and let $\lambda_0 \in \mathbb{F}$. Then the following are equivalent:

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- For a field $\mathbb{F}$, a matrix $A \in \mathbb{F}^{n \times n}$, and an eigenvalue λ_0 of A, the *algebraic multiplicity* of the eigenvalue λ_0 is the largest integer k s.t. $(\lambda - \lambda_0)^k | p_A(\lambda)$.

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- For a field $\mathbb{F}$, a matrix $A \in \mathbb{F}^{n\times n}$, and an eigenvalue λ_0 of A, the *algebraic multiplicity* of the eigenvalue λ_0 is the largest integer k s.t. $(\lambda - \lambda_0)^k | p_A(\lambda)$.
- Note that the sum of algebraic multiplicities of the matrix $A \in \mathbb{F}^{n\times n}$ is at most n.
  - If the field $\mathbb{F}$ is algebraically closed, then the sum of algebraic multiplicities of A is exactly n.
  - However, if $\mathbb{F}$ is not algebraically closed, then it may possibly be strictly less than n.

Theorem 2.5

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Then the geometric multiplicity of any eigenvalue of A is no greater than the algebraic multiplicity of that eigenvalue.

Proof. Omitted.

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Proof. Omitted.

- Schematically, for an eigenvalue λ of a matrix $A \in \mathbb{F}^{n \times n}$ (where $\mathbb{F}$ is an arbitrary field), Theorem 2.5 states that:

$$\text{geometric multiplicity of } \lambda \quad \leq \quad \text{algebraic multiplicity of } \lambda.$$

Example 2.6

Consider the matrix

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

in $\mathbb{C}^{3\times 3}$.

(a) Compute the characteristic polynomial $p_A(\lambda)$ of the matrix A.

(b) Find all the eigenvalues of A and their algebraic multiplicities.

(c) For each eigenvalue λ of A, find a basis for the eigenspace E_λ and specify the geometric multiplicity of the eigenvalue λ.

Solution. (a) The characteristic polynomial of A is:

$$\begin{aligned} p_A(\lambda) &= \det(\lambda I_3 - A) \\ &= \begin{vmatrix} \lambda - 4 & 0 & 2 \\ -2 & \lambda - 5 & -4 \\ 0 & 0 & \lambda - 5 \end{vmatrix} \\ &\overset{(*)}{=} (\lambda - 4)(\lambda - 5)^2 \\ &= \lambda^3 - 14\lambda^2 + 65\lambda - 100. \end{aligned}$$

where the easiest way to obtain (*) is via Laplace expansion along the second column.

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where the easiest way to obtain (*) is via Laplace expansion along the second column.

- **Remark:** We did not really need to expand in the last line. We only really care about the roots of the characteristic polynomial, and it is more convenient to have a form that is already factored.)

Solution (continued). Reminder: (a) $p_A(\lambda) = (\lambda - 4)(\lambda - 5)^2$.

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(b) From part (a), we see that A has two eigenvalues, namely, the eigenvalue $\lambda_1 = 4$ (with algebraic multiplicity 1), and the eigenvalue $\lambda_2 = 5$ (with algebraic multiplicity 2).

Solution (continued). Reminder: (a) $p_A(\lambda) = (\lambda - 4)(\lambda - 5)^2$.

(b) From part (a), we see that A has two eigenvalues, namely, the eigenvalue $\lambda_1 = 4$ (with algebraic multiplicity 1), and the eigenvalue $\lambda_2 = 5$ (with algebraic multiplicity 2).

(c) For each $i \in \{1, 2\}$, the eigenspace E_{λ_i} is precisely the set of solutions of the equation $A\mathbf{x} = \lambda_i \mathbf{x}$, which is obviously equivalent to the equation

$$(\lambda_i I_3 - A)\mathbf{x} \quad = \quad \mathbf{0}.$$

Solution (continued). For $\lambda_1 = 4$, we have that

$$\lambda_1 I_3 - A = \begin{bmatrix} \lambda_1 - 4 & 0 & 2 \\ -2 & \lambda_1 - 5 & -4 \\ 0 & 0 & \lambda_1 - 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ -2 & -1 & -4 \\ 0 & 0 & -1 \end{bmatrix},$$

and that

$$\text{RREF}(\lambda_1 I_3 - A) = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution (continued). For $\lambda_1 = 4$, we have that

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and that

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Thus, the general solution of the equation $(\lambda_1 I_3 - A)\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} -t/2 \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} = \tfrac{t}{2}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \qquad \text{with } t \in \mathbb{C}.$$

So, $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ is a basis of the eigespace E_{λ_1}, and we see that the eigenvalue $\lambda_1 = 4$ has geometric multiplicity 1.

Solution (continued). For $\lambda_2 = 5$, we have that

$$\lambda_2 I_3 - A = \begin{bmatrix} \lambda_2 - 4 & 0 & 2 \\ -2 & \lambda_2 - 5 & -4 \\ 0 & 0 & \lambda_2 - 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix},$$

and that

$$\mathrm{RREF}(\lambda_2 I_3 - A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the general solution of the equation $(\lambda_1 I_3 - A)\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} -2t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{with } s, t \in \mathbb{C}.$$

So, $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of the eigenspace E_{λ_2}, and we see that the eigenvalue $\lambda_2 = 5$ has geometric multiplicity 2.

Proposition 2.9

Let $\mathbb{F}$ be a field, and let A be a triangular matrix in $\mathbb{F}^{n \times n}$. Then the eigenvalues of A are precisely the entries of A on its main diagonal, and moreover, the algebraic multiplicity of each eigenvalue is precisely the number of times that it appears on the main diagonal of A.[a]

[a] However, the geometric multiplicity may possibly be smaller, as Example 2.10 shows.

Proof.

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Let $\mathbb{F}$ be a field, and let A be a triangular matrix in $\mathbb{F}^{n \times n}$. Then the eigenvalues of A are precisely the entries of A on its main diagonal, and moreover, the algebraic multiplicity of each eigenvalue is precisely the number of times that it appears on the main diagonal of A.[a]

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Proof. Set $A = [\ a_{i,j}\ ]_{n \times n}$.

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Proof. Set $A = [\ a_{i,j}\ ]_{n\times n}$. Since A is triangular, so is the matrix $\lambda I_n - A$; so, the determinant of $\lambda I_n - A$ can be computed simply by multiplying its entries on the main diagonal.

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Proof. Set $A = [\ a_{i,j}\ ]_{n\times n}$. Since A is triangular, so is the matrix $\lambda I_n - A$; so, the determinant of $\lambda I_n - A$ can be computed simply by multiplying its entries on the main diagonal. It follows that the characteristic polynomial of A is

$$p_A(\lambda) \;=\; \det(\lambda I_n - A) \;=\; (\lambda - a_{1,1})(\lambda - a_{2,2})\dots(\lambda - a_{n,n}),$$

and the result follows. Q.E.D.

Example 2.10

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

in $\mathbb{C}^{5\times 5}$.

a. Compute the characteristic polynomial $p_A(\lambda)$ of the matrix A.
b. Find all the eigenvalues of A and their algebraic multiplicities.
c. For each eigenvalue λ of A, find a basis for the eigenspace E_λ and specify the geometric multiplicity of the eigenvalue λ.

Solution. (a) The matrix A is upper triangular, and so its characteristic polynomial is

$$\begin{aligned} p_A(\lambda) &= \det(\lambda I_5 - A) \\ &= \begin{vmatrix} \lambda-1 & -2 & 0 & 0 & 0 \\ 0 & \lambda-2 & 0 & 0 & 0 \\ 0 & 0 & \lambda-1 & -1 & -3 \\ 0 & 0 & 0 & \lambda-3 & -3 \\ 0 & 0 & 0 & 0 & \lambda-3 \end{vmatrix} \\ &= (\lambda-1)^2(\lambda-2)(\lambda-3)^2. \end{aligned}$$

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We see from part (a) that A has three eigenvalues, namely, $\lambda_1 = 1$ (with algebraic multiplicity 2), $\lambda_2 = 2$ (with algebraic multiplicity 1), and $\lambda = 3$ (with algebraic multiplicity 2).

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(c) For each $i \in \{1, 2, 3\}$, the eigenspace E_{λ_i} is precisely the set of solutions of the equation $A\mathbf{x} = \lambda_i\mathbf{x}$, which is obviously equivalent to the equation

$$(\lambda_i I_5 - A)\mathbf{x} = \mathbf{0}.$$

Solution (continued). For $\lambda_1 = 1$, we have that

$$\lambda_1 I_5 - A = \begin{bmatrix} 0 & -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix},$$

and that

$$\text{RREF}(\lambda_1 I_5 - A) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

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Thus, the general solution of the equation $(\lambda_1 I_5 - A)\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} s \\ 0 \\ t \\ 0 \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{with } s, t \in \mathbb{C}.$$

Solution (continued). For $\lambda_1 = 1$ (continued):

So,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

is a basis of the eigenspace E_{λ_1}, and we see that the eigenvalue $\lambda_1 = 1$ has geometric multiplicity 2.

Solution (continued). For $\lambda_2 = 2$, we have that

$$\lambda_2 I_5 - A = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

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Solution (continued). For $\lambda_2 = 2$, we have that

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Thus, the general solution of the equation $(\lambda_2 I_5 - A)\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} 2t \\ t \\ 0 \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \text{with } t \in \mathbb{C}.$$

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is a basis of the eigenspace E_{λ_2}, and we see that the eigenvalue $\lambda_2 = 2$ has geometric multiplicity 1.

Solution (continued). For $\lambda_3 = 3$, we have that

$$\lambda_3 I_5 - A = \begin{bmatrix} 2 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & -3 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and that

$$\text{RREF}(\lambda_3 I_5 - A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the general solution of the equation $(\lambda_3 I_5 - A)\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \frac{t}{2} \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} = \frac{t}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \qquad \text{with } t \in \mathbb{C}.$$

Solution (continued). For $\lambda_3 = 3$ (continued):

So,

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

is a basis of the eigenspace E_{λ_3}, and we see that the eigenvalue $\lambda_3 = 3$ has geometric multiplicity 1.

Definition

Given a field $\mathbb{F}$ and a matrix $A \in \mathbb{F}^{n \times n}$, the *spectrum* of A is the multiset of all eigenvalues of A, and the number of times that each eigenvalue appears in the spectrum is precisely equal to the algebraic multiplicity of that eigenvalue.

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- For example, if $A \in \mathbb{C}^{5 \times 5}$ has eigenvalues 1 (with algeraic multiplicity 1), $1 + i$ (with algebraic multiplicity 2), and $1 - i$ (with algebraic multiplicity 2), then the spectrum of A is $\{1, 1+i, 1+i, 1-i, 1-i\}$.

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For a matrix $A \in \mathbb{C}^{n\times n}$, the *spectral radius* of A, denoted by $\rho(A)$, is the maximum absolute value of any eigenvalue of A.

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- For example, if the spectrum of a matrix $A \in \mathbb{C}^{5 \times 5}$ is $\{1, 1 + i, 1 + i, 1 - i, 1 - i\}$, then the spectral radius of A is $\rho(A) = \max\{|1|, |1 + i|, |1 + i|, |1 - i|, |1 - i|\} = \sqrt{2}$.

Theorem 1.1

Let $p(x)$ be any polynomial with real coefficients, and let $z \in \mathbb{C}$. Then z is a root of $p(x)$ iff $\overline{z}$ is a root of $p(x)$.

Theorem 2.4

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times n}$, and let $\lambda_0 \in \mathbb{F}$. Then the following are equivalent:

1. (i) λ_0 is an eigenvalue of A;
2. (ii) λ_0 is a root of the characteristic polynomial of A, i.e. $p_A(\lambda_0) = 0$;
3. (iii) λ_0 is a solution of the characteristic equation of A, i.e. $\det(\lambda_0 I_n - A) = 0$.

- In view of Theorems 1.1 and 2.4, we can visualize the complex eigenvalues of an $n \times n$ matrix A with **real** entries.
    - However, we consider A to be a matrix in the vector space $\mathbb{C}^{n \times n}$, so that it can have complex eigenvalues.

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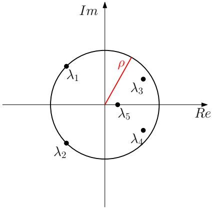
- Its characteristic polynomial $p_A(\lambda)$ is of degree n and has real coefficients.
- By Theorem 1.1, the roots of this polynomial come in conjugate pairs, and moreover, by Theorem 2.4, those roots are precisely the eigenvalues of A.
- The eigenvalues all lie in the complex plane, in the disk centered at the origin and with radius $\rho(A)$, and they are symmetric about the real axis.



Theorem 2.11

Let $\mathbb{F}$ be a field, let $A = [\ a_{i,j}\ ]$ be a matrix in $\mathbb{F}^{n\times n}$, and assume that $\{\lambda_1, \ldots, \lambda_n\}$ is the spectrum of A. Then

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- **Remark:** Theorem 2.11 only applies if the spectrum of the matrix $A \in \mathbb{F}^{n\times n}$ contains n eigenvalues (including algebraic multiplicities)!
  - This will always happen if $\mathbb{F}$ is algebraically closed, but need not happen otherwise.

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(a) By setting $\lambda = 0$, we obtain

$$p_A(0) \quad = \quad (0 - \lambda_1) \dots (0 - \lambda_n) \quad = \quad (-1)^n \lambda_1 \dots \lambda_n.$$

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We now deduce that $(-1)^n \lambda_1 \dots \lambda_n = (-1)^n \det(A)$, and it follows that $\det(A) = \lambda_1 \dots \lambda_n$.

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(b) We will compute the coefficient in front of λ^{n-1} in the characteristic polynomial $p_A(\lambda)$ in two ways.

First, since $p_A(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n)$, it is clear that the coefficient in front of λ^{n-1} is $-\lambda_1 - \dots - \lambda_n$.

Solution (continued). Reminder: WTS $\text{trace}(A) = \lambda_1 + \cdots + \lambda_n$; we are computing the coefficient in front of λ^{n-1} in $p_A(\lambda)$.

On the other hand, we have that

$$p_A(\lambda) \quad = \quad \det(\lambda I_n - A) \quad = \quad \begin{vmatrix} \lambda - a_{1,1} & -a_{1,2} & \dots & -a_{1,n} \\ -a_{2,1} & \lambda - a_{2,2} & \dots & -a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n,1} & -a_{n,2} & \dots & \lambda - a_{n,n} \end{vmatrix}.$$

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We now use the definition of the determinant: the only permutation $\sigma \in S_n$ that produces a polynomial with λ^{n-1} appearing with it (with a possibly non-zero coefficient) is the identity permutation
$\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}$,

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Solution (continued). Reminder: WTS $\text{trace}(A) = \lambda_1 + \cdots + \lambda_n$; we are computing the coefficient in front of λ^{n-1} in $p_A(\lambda)$.

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Let $\mathbb{F}$ be a field, let $A = [\ a_{i,j}\ ]$ be a matrix in $\mathbb{F}^{n\times n}$, and assume that $\{\lambda_1, \ldots, \lambda_n\}$ is the spectrum of A. Then

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Theorem 2.12

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times n}$, and let $\lambda_1, \ldots, \lambda_k$ be pairwise distinct eigenvalues of A. For each $i \in \{1, \ldots, k\}$, let $\mathcal{B}_{\lambda_i} = \{\mathbf{v}_1^{\lambda_i}, \ldots, \mathbf{v}_{t_i}^{\lambda_i}\}$ be a basis of the eigenspace E_{λ_i}. Then

$$\{\mathbf{v}_1^{\lambda_1}, \ldots, \mathbf{v}_{t_1}^{\lambda_1}, \mathbf{v}_1^{\lambda_2}, \ldots, \mathbf{v}_{t_2}^{\lambda_2}, \ldots, \mathbf{v}_1^{\lambda_k}, \ldots, \mathbf{v}_{t_k}^{\lambda_k}\}$$

is a linearly independent set of vectors in $\mathbb{F}^n$.

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Claim. *For all $\mathbf{v}_1 \in E_{\lambda_1} \setminus \{\mathbf{0}\}, \dots, \mathbf{v}_k \in E_{\lambda_k} \setminus \{\mathbf{0}\}$, the set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent.*

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Proof of the Claim. Suppose otherwise, and consider an inclusion-wise minimal subset of $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ that is linearly dependent. After possibly permuting the order of our eigenvalues, WMA $\exists \ell \in \{1, \dots, k\}$ s.t. $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ is an inclusion-wise minimal linearly dependent set, i.e. it is linearly dependent, but all its proper subsets are linearly independent.

Claim. *For all* $\mathbf{v}_1 \in E_{\lambda_1} \setminus \{\mathbf{0}\}, \dots, \mathbf{v}_k \in E_{\lambda_k} \setminus \{\mathbf{0}\}$*, the set* $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ *is linearly independent.*

Proof of the Claim (continued). Reminder: $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ is an inclusion-wise minimal linearly dependent set.

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Claim. *For all* $\mathbf{v}_1 \in E_{\lambda_1} \setminus \{\mathbf{0}\}, \dots, \mathbf{v}_k \in E_{\lambda_k} \setminus \{\mathbf{0}\}$, *the set* $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ *is linearly independent.*

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So, $\alpha_1\lambda_1\mathbf{v}_1 + \dots + \alpha_\ell\lambda_\ell\mathbf{v}_\ell = \alpha_1\lambda_\ell\mathbf{v}_1 + \dots + \alpha_\ell\lambda_\ell\mathbf{v}_\ell$, and consequently,

$$\alpha_1(\lambda_1 - \lambda_\ell)\mathbf{v}_1 + \dots + \alpha_{\ell-1}(\lambda_{\ell-1} - \lambda_\ell)\mathbf{v}_{\ell-1} = \mathbf{0}.$$

Since $\alpha_1, \dots, \alpha_{\ell-1}$ are all non-zero, and since $\lambda_1, \dots, \lambda_\ell$ are pairwise distinct, we see that the scalars $\alpha_1(\lambda_1 - \lambda_\ell), \dots, \alpha_{\ell-1}(\lambda_{\ell-1} - \lambda_\ell)$ are all non-zero.

Claim. *For all* $\mathbf{v}_1 \in E_{\lambda_1} \setminus \{\mathbf{0}\}, \dots, \mathbf{v}_k \in E_{\lambda_k} \setminus \{\mathbf{0}\}$*, the set* $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ *is linearly independent.*

Proof of the Claim (continued). Reminder: $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ is an inclusion-wise minimal linearly dependent set.

So, $\alpha_1\lambda_1\mathbf{v}_1 + \cdots + \alpha_\ell\lambda_\ell\mathbf{v}_\ell = \alpha_1\lambda_\ell\mathbf{v}_1 + \cdots + \alpha_\ell\lambda_\ell\mathbf{v}_\ell$, and consequently,

$$\alpha_1(\lambda_1 - \lambda_\ell)\mathbf{v}_1 + \cdots + \alpha_{\ell-1}(\lambda_{\ell-1} - \lambda_\ell)\mathbf{v}_{\ell-1} \quad = \quad \mathbf{0}.$$

Since $\alpha_1, \dots, \alpha_{\ell-1}$ are all non-zero, and since $\lambda_1, \dots, \lambda_\ell$ are pairwise distinct, we see that the scalars $\alpha_1(\lambda_1 - \lambda_\ell), \dots, \alpha_{\ell-1}(\lambda_{\ell-1} - \lambda_\ell)$ are all non-zero. So, $\{\mathbf{v}_1, \dots, \mathbf{v}_{\ell-1}\}$ is linearly dependent, contrary to the minimality of $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$. This proves the Claim.

Theorem 2.12

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times n}$, and let $\lambda_1, \dots, \lambda_k$ be pairwise distinct eigenvalues of A. For each $i \in \{1, \dots, k\}$, let $\mathcal{B}_{\lambda_i} = \{\mathbf{v}_1^{\lambda_i}, \dots, \mathbf{v}_{t_i}^{\lambda_i}\}$ be a basis of the eigenspace E_{λ_i}. Then

$$\{\mathbf{v}_1^{\lambda_1}, \dots, \mathbf{v}_{t_1}^{\lambda_1}, \mathbf{v}_1^{\lambda_2}, \dots, \mathbf{v}_{t_2}^{\lambda_2}, \dots, \mathbf{v}_1^{\lambda_k}, \dots, \mathbf{v}_{t_k}^{\lambda_k}\}$$

is a linearly independent set of vectors in $\mathbb{F}^n$.

Proof (continued). Now, suppose that the set

$$\{\mathbf{v}_1^{\lambda_1}, \dots, \mathbf{v}_{t_1}^{\lambda_1}, \mathbf{v}_1^{\lambda_2}, \dots, \mathbf{v}_{t_2}^{\lambda_2}, \dots, \mathbf{v}_1^{\lambda_k}, \dots, \mathbf{v}_{t_k}^{\lambda_k}\}$$

is linearly dependent.

Theorem 2.12

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times n}$, and let $\lambda_1, \dots, \lambda_k$ be pairwise distinct eigenvalues of A. For each $i \in \{1, \dots, k\}$, let $\mathcal{B}_{\lambda_i} = \{\mathbf{v}_1^{\lambda_i}, \dots, \mathbf{v}_{t_i}^{\lambda_i}\}$ be a basis of the eigenspace E_{λ_i}. Then

$$\{\mathbf{v}_1^{\lambda_1}, \dots, \mathbf{v}_{t_1}^{\lambda_1}, \mathbf{v}_1^{\lambda_2}, \dots, \mathbf{v}_{t_2}^{\lambda_2}, \dots, \mathbf{v}_1^{\lambda_k}, \dots, \mathbf{v}_{t_k}^{\lambda_k}\}$$

is a linearly independent set of vectors in $\mathbb{F}^n$.

Proof (continued). Now, suppose that the set

$$\{\mathbf{v}_1^{\lambda_1}, \dots, \mathbf{v}_{t_1}^{\lambda_1}, \mathbf{v}_1^{\lambda_2}, \dots, \mathbf{v}_{t_2}^{\lambda_2}, \dots, \mathbf{v}_1^{\lambda_k}, \dots, \mathbf{v}_{t_k}^{\lambda_k}\}$$

is linearly dependent. Then there exist scalars

$$\alpha_1^{\lambda_1}, \dots, \alpha_{t_1}^{\lambda_1}, \alpha_1^{\lambda_2}, \dots, \alpha_{t_2}^{\lambda_2}, \dots, \alpha_1^{\lambda_k}, \dots, \alpha_{t_k}^{\lambda_k} \in \mathbb{F},$$

not all zero, s.t. $\sum_{i=1}^{k} \sum_{j=1}^{t_i} \alpha_j^{\lambda_i} \mathbf{v}_j^{\lambda_i} = \mathbf{0}$.

Theorem 2.12

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$$\{\mathbf{v}_1^{\lambda_1}, \ldots, \mathbf{v}_{t_1}^{\lambda_1}, \mathbf{v}_1^{\lambda_2}, \ldots, \mathbf{v}_{t_2}^{\lambda_2}, \ldots, \mathbf{v}_1^{\lambda_k}, \ldots, \mathbf{v}_{t_k}^{\lambda_k}\}$$

is a linearly independent set of vectors in $\mathbb{F}^n$.

Proof (continued). Reminder: $\sum_{i=1}^{k} \sum_{j=1}^{t_i} \alpha_j^{\lambda_i} \mathbf{v}_j^{\lambda_i} = \mathbf{0}$.

Theorem 2.12

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times n}$, and let $\lambda_1, \dots, \lambda_k$ be pairwise distinct eigenvalues of A. For each $i \in \{1, \dots, k\}$, let $\mathcal{B}_{\lambda_i} = \{\mathbf{v}_1^{\lambda_i}, \dots, \mathbf{v}_{t_i}^{\lambda_i}\}$ be a basis of the eigenspace E_{λ_i}. Then

$$\{\mathbf{v}_1^{\lambda_1}, \dots, \mathbf{v}_{t_1}^{\lambda_1}, \mathbf{v}_1^{\lambda_2}, \dots, \mathbf{v}_{t_2}^{\lambda_2}, \dots, \mathbf{v}_1^{\lambda_k}, \dots, \mathbf{v}_{t_k}^{\lambda_k}\}$$

is a linearly independent set of vectors in $\mathbb{F}^n$.

Proof (continued). Reminder: $\sum_{i=1}^{k} \sum_{j=1}^{t_i} \alpha_j^{\lambda_i} \mathbf{v}_j^{\lambda_i} = \mathbf{0}$.

For each $i \in \{1, \dots, k\}$, set $\mathbf{v}_i := \sum_{j=1}^{t_i} \alpha_j^{\lambda_i} \mathbf{v}_j^{\lambda_i}$;

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$$\{\mathbf{v}_1^{\lambda_1}, \dots, \mathbf{v}_{t_1}^{\lambda_1}, \mathbf{v}_1^{\lambda_2}, \dots, \mathbf{v}_{t_2}^{\lambda_2}, \dots, \mathbf{v}_1^{\lambda_k}, \dots, \mathbf{v}_{t_k}^{\lambda_k}\}$$

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Proof (continued). Reminder: $\sum_{i=1}^{k} \sum_{j=1}^{t_i} \alpha_j^{\lambda_i} \mathbf{v}_j^{\lambda_i} = \mathbf{0}$.

For each $i \in \{1, \dots, k\}$, set $\mathbf{v}_i := \sum_{j=1}^{t_i} \alpha_j^{\lambda_i} \mathbf{v}_j^{\lambda_i}$; then $\mathbf{v}_i \in E_{\lambda_i}$, and moreover, since $\{\mathbf{v}_1^{\lambda_i}, \dots, \mathbf{v}_{t_i}^{\lambda_i}\}$ is linearly independent, we see that $\mathbf{v}_i = \mathbf{0}$ iff $\alpha_1^{\lambda_i} = \dots = \alpha_{t_i}^{\lambda_i} = 0$.

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Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times n}$, and let $\lambda_1, \dots, \lambda_k$ be pairwise distinct eigenvalues of A. For each $i \in \{1, \dots, k\}$, let $\mathcal{B}_{\lambda_i} = \{\mathbf{v}_1^{\lambda_i}, \dots, \mathbf{v}_{t_i}^{\lambda_i}\}$ be a basis of the eigenspace E_{λ_i}. Then

$$\{\mathbf{v}_1^{\lambda_1}, \dots, \mathbf{v}_{t_1}^{\lambda_1}, \mathbf{v}_1^{\lambda_2}, \dots, \mathbf{v}_{t_2}^{\lambda_2}, \dots, \mathbf{v}_1^{\lambda_k}, \dots, \mathbf{v}_{t_k}^{\lambda_k}\}$$

is a linearly independent set of vectors in $\mathbb{F}^n$.

Proof (continued). Reminder: $\sum_{i=1}^{k} \sum_{j=1}^{t_i} \alpha_j^{\lambda_i} \mathbf{v}_j^{\lambda_i} = \mathbf{0}$.

For each $i \in \{1, \dots, k\}$, set $\mathbf{v}_i := \sum_{j=1}^{t_i} \alpha_j^{\lambda_i} \mathbf{v}_j^{\lambda_i}$; then $\mathbf{v}_i \in E_{\lambda_i}$, and moreover, since $\{\mathbf{v}_1^{\lambda_i}, \dots, \mathbf{v}_{t_i}^{\lambda_i}\}$ is linearly independent, we see that $\mathbf{v}_i = \mathbf{0}$ iff $\alpha_1^{\lambda_i} = \dots = \alpha_{t_i}^{\lambda_i} = 0$. Since not all $\alpha_j^{\lambda_i}$'s are zero, we see that at least one of $\mathbf{v}_1, \dots, \mathbf{v}_k$ is non-zero.

Theorem 2.12

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times n}$, and let $\lambda_1, \dots, \lambda_k$ be pairwise distinct eigenvalues of A. For each $i \in \{1, \dots, k\}$, let $\mathcal{B}_{\lambda_i} = \{\mathbf{v}_1^{\lambda_i}, \dots, \mathbf{v}_{t_i}^{\lambda_i}\}$ be a basis of the eigenspace E_{λ_i}. Then

$$\{\mathbf{v}_1^{\lambda_1}, \dots, \mathbf{v}_{t_1}^{\lambda_1}, \mathbf{v}_1^{\lambda_2}, \dots, \mathbf{v}_{t_2}^{\lambda_2}, \dots, \mathbf{v}_1^{\lambda_k}, \dots, \mathbf{v}_{t_k}^{\lambda_k}\}$$

is a linearly independent set of vectors in $\mathbb{F}^n$.

Proof (continued). Reminder: $\sum_{i=1}^{k} \sum_{j=1}^{t_i} \alpha_j^{\lambda_i} \mathbf{v}_j^{\lambda_i} = \mathbf{0}$.

For each $i \in \{1, \dots, k\}$, set $\mathbf{v}_i := \sum_{j=1}^{t_i} \alpha_j^{\lambda_i} \mathbf{v}_j^{\lambda_i}$; then $\mathbf{v}_i \in E_{\lambda_i}$, and moreover, since $\{\mathbf{v}_1^{\lambda_i}, \dots, \mathbf{v}_{t_i}^{\lambda_i}\}$ is linearly independent, we see that $\mathbf{v}_i = \mathbf{0}$ iff $\alpha_1^{\lambda_i} = \dots = \alpha_{t_i}^{\lambda_i} = 0$. Since not all $\alpha_j^{\lambda_i}$'s are zero, we see that at least one of $\mathbf{v}_1, \dots, \mathbf{v}_k$ is non-zero. After possibly permuting the order of our eigenvalues, WMA $\exists \ell \in \{1, \dots, k\}$ s.t. $\mathbf{v}_1, \dots, \mathbf{v}_\ell$ are all non-zero, whereas $\mathbf{v}_{\ell+1} = \dots = \mathbf{v}_k = \mathbf{0}$.

Theorem 2.12

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times n}$, and let $\lambda_1, \dots, \lambda_k$ be pairwise distinct eigenvalues of A. For each $i \in \{1, \dots, k\}$, let $\mathcal{B}_{\lambda_i} = \{\mathbf{v}_1^{\lambda_i}, \dots, \mathbf{v}_{t_i}^{\lambda_i}\}$ be a basis of the eigenspace E_{λ_i}. Then

$$\{\mathbf{v}_1^{\lambda_1}, \dots, \mathbf{v}_{t_1}^{\lambda_1}, \mathbf{v}_1^{\lambda_2}, \dots, \mathbf{v}_{t_2}^{\lambda_2}, \dots, \mathbf{v}_1^{\lambda_k}, \dots, \mathbf{v}_{t_k}^{\lambda_k}\}$$

is a linearly independent set of vectors in $\mathbb{F}^n$.

Proof (continued). Reminder: $\sum_{i=1}^{k} \sum_{j=1}^{t_i} \alpha_j^{\lambda_i} \mathbf{v}_j^{\lambda_i} = \mathbf{0}$.

For each $i \in \{1, \dots, k\}$, set $\mathbf{v}_i := \sum_{j=1}^{t_i} \alpha_j^{\lambda_i} \mathbf{v}_j^{\lambda_i}$; then $\mathbf{v}_i \in E_{\lambda_i}$, and moreover, since $\{\mathbf{v}_1^{\lambda_i}, \dots, \mathbf{v}_{t_i}^{\lambda_i}\}$ is linearly independent, we see that $\mathbf{v}_i = \mathbf{0}$ iff $\alpha_1^{\lambda_i} = \cdots = \alpha_{t_i}^{\lambda_i} = 0$. Since not all $\alpha_j^{\lambda_i}$'s are zero, we see that at least one of $\mathbf{v}_1, \dots, \mathbf{v}_k$ is non-zero. After possibly permuting the order of our eigenvalues, WMA $\exists \ell \in \{1, \dots, k\}$ s.t. $\mathbf{v}_1, \dots, \mathbf{v}_\ell$ are all non-zero, whereas $\mathbf{v}_{\ell+1} = \cdots = \mathbf{v}_k = \mathbf{0}$. Then (next slide):

Theorem 2.12

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times n}$, and let $\lambda_1, \dots, \lambda_k$ be pairwise distinct eigenvalues of A. For each $i \in \{1, \dots, k\}$, let $\mathcal{B}_{\lambda_i} = \{\mathbf{v}_1^{\lambda_i}, \dots, \mathbf{v}_{t_i}^{\lambda_i}\}$ be a basis of the eigenspace E_{λ_i}. Then

$$\{\mathbf{v}_1^{\lambda_1}, \dots, \mathbf{v}_{t_1}^{\lambda_1}, \mathbf{v}_1^{\lambda_2}, \dots, \mathbf{v}_{t_2}^{\lambda_2}, \dots, \mathbf{v}_1^{\lambda_k}, \dots, \mathbf{v}_{t_k}^{\lambda_k}\}$$

is a linearly independent set of vectors in $\mathbb{F}^n$.

Proof (continued).

Claim. *For all* $\mathbf{v}_1 \in E_{\lambda_1} \setminus \{\mathbf{0}\}, \dots, \mathbf{v}_k \in E_{\lambda_k} \setminus \{\mathbf{0}\}$*, the set* $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ *is linearly independent.*

$$\mathbf{v}_1 + \dots + \mathbf{v}_\ell \;=\; \mathbf{v}_1 + \dots + \mathbf{v}_k \;=\; \sum_{i=1}^{k} \sum_{j=1}^{t_i} \alpha_j^{\lambda_i} \mathbf{v}_j^{\lambda_i} \;=\; \mathbf{0}.$$

Theorem 2.12

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times n}$, and let $\lambda_1, \dots, \lambda_k$ be pairwise distinct eigenvalues of A. For each $i \in \{1, \dots, k\}$, let $\mathcal{B}_{\lambda_i} = \{\mathbf{v}_1^{\lambda_i}, \dots, \mathbf{v}_{t_i}^{\lambda_i}\}$ be a basis of the eigenspace E_{λ_i}. Then

$$\{\mathbf{v}_1^{\lambda_1}, \dots, \mathbf{v}_{t_1}^{\lambda_1}, \mathbf{v}_1^{\lambda_2}, \dots, \mathbf{v}_{t_2}^{\lambda_2}, \dots, \mathbf{v}_1^{\lambda_k}, \dots, \mathbf{v}_{t_k}^{\lambda_k}\}$$

is a linearly independent set of vectors in $\mathbb{F}^n$.

Proof (continued).

Claim. *For all $\mathbf{v}_1 \in E_{\lambda_1} \setminus \{\mathbf{0}\}, \dots, \mathbf{v}_k \in E_{\lambda_k} \setminus \{\mathbf{0}\}$, the set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent.*

$$\mathbf{v}_1 + \dots + \mathbf{v}_\ell \;=\; \mathbf{v}_1 + \dots + \mathbf{v}_k \;=\; \sum_{i=1}^{k} \sum_{j=1}^{t_i} \alpha_j^{\lambda_i} \mathbf{v}_j^{\lambda_i} \;=\; \mathbf{0}.$$

It follows that $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ is linearly dependent.

Theorem 2.12

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times n}$, and let $\lambda_1, \dots, \lambda_k$ be pairwise distinct eigenvalues of A. For each $i \in \{1, \dots, k\}$, let $\mathcal{B}_{\lambda_i} = \{\mathbf{v}_1^{\lambda_i}, \dots, \mathbf{v}_{t_i}^{\lambda_i}\}$ be a basis of the eigenspace E_{λ_i}. Then

$$\{\mathbf{v}_1^{\lambda_1}, \dots, \mathbf{v}_{t_1}^{\lambda_1}, \mathbf{v}_1^{\lambda_2}, \dots, \mathbf{v}_{t_2}^{\lambda_2}, \dots, \mathbf{v}_1^{\lambda_k}, \dots, \mathbf{v}_{t_k}^{\lambda_k}\}$$

is a linearly independent set of vectors in $\mathbb{F}^n$.

Proof (continued).

Claim. *For all* $\mathbf{v}_1 \in E_{\lambda_1} \setminus \{\mathbf{0}\}, \dots, \mathbf{v}_k \in E_{\lambda_k} \setminus \{\mathbf{0}\}$, *the set* $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ *is linearly independent.*

$$\mathbf{v}_1 + \dots + \mathbf{v}_\ell \quad = \quad \mathbf{v}_1 + \dots + \mathbf{v}_k \quad = \quad \sum_{i=1}^{k} \sum_{j=1}^{t_i} \alpha_j^{\lambda_i} \mathbf{v}_j^{\lambda_i} \quad = \quad \mathbf{0}.$$

It follows that $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ is linearly dependent. But since $\mathbf{v}_1 \in E_{\lambda_1} \setminus \{\mathbf{0}\}, \dots, \mathbf{v}_\ell \in E_{\lambda_\ell} \setminus \{\mathbf{0}\}$, this contradicts the Claim. Q.E.D.

Corollary 2.13

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Then the following are equivalent:

1. (i) $\mathbb{F}^n$ has a basis formed by eigenvectors of A;
2. (ii) the sum of algebraic multiplicities of all distinct eigenvalues A is equal to n, and the geometric multiplicity of each eigenvalue of A is equal to the algebraic multiplicity of that eigenvalue;
3. (iii) the sum of geometric multiplicities of all distinct eigenvalues A is equal to n.

Proof.

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Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Then the following are equivalent:

(i) $\mathbb{F}^n$ has a basis formed by eigenvectors of A;

(ii) the sum of algebraic multiplicities of all distinct eigenvalues A is equal to n, and the geometric multiplicity of each eigenvalue of A is equal to the algebraic multiplicity of that eigenvalue;

(iii) the sum of geometric multiplicities of all distinct eigenvalues A is equal to n.

Proof. Obviously, (ii) implies (iii).

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(iii) the sum of geometric multiplicities of all distinct eigenvalues A is equal to n.

Proof. Obviously, (ii) implies (iii). The fact that (iii) implies (ii) follows from the fact that the sum of algebraic multiplicities of the eigenvalues of A is at most n, and the fact that (by Theorem 2.5) the geometric multiplicity of any eigenvalue of A is no greater than the algebraic multiplicity of that eigenvalue.

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Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Then the following are equivalent:

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(iii) the sum of geometric multiplicities of all distinct eigenvalues A is equal to n.

Proof. Obviously, (ii) implies (iii). The fact that (iii) implies (ii) follows from the fact that the sum of algebraic multiplicities of the eigenvalues of A is at most n, and the fact that (by Theorem 2.5) the geometric multiplicity of any eigenvalue of A is no greater than the algebraic multiplicity of that eigenvalue.

It now suffices to show that (i) and (iii) are equivalent.

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Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Then the following are equivalent:

1. (i) $\mathbb{F}^n$ has a basis formed by eigenvectors of A;
2. (ii) the sum of algebraic multiplicities of all distinct eigenvalues A is equal to n, and the geometric multiplicity of each eigenvalue of A is equal to the algebraic multiplicity of that eigenvalue;
3. (iii) the sum of geometric multiplicities of all distinct eigenvalues A is equal to n.

Proof (continued). Reminder: WTS "(i) $\iff$ (iii)."

Corollary 2.13

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$. Then the following are equivalent:

1. $\mathbb{F}^n$ has a basis formed by eigenvectors of A;
2. the sum of algebraic multiplicities of all distinct eigenvalues A is equal to n, and the geometric multiplicity of each eigenvalue of A is equal to the algebraic multiplicity of that eigenvalue;
3. the sum of geometric multiplicities of all distinct eigenvalues A is equal to n.

Proof (continued). Reminder: WTS "(i) $\iff$ (iii)."

If (i) holds, then we see that the sum of geometric multiplicities of all distinct eigenvalues of A must be at least n;

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2. (ii) the sum of algebraic multiplicities of all distinct eigenvalues A is equal to n, and the geometric multiplicity of each eigenvalue of A is equal to the algebraic multiplicity of that eigenvalue;
3. (iii) the sum of geometric multiplicities of all distinct eigenvalues A is equal to n.

Proof (continued). Reminder: WTS "(i) $\iff$ (iii)."

If (i) holds, then we see that the sum of geometric multiplicities of all distinct eigenvalues of A must be at least n; since this sum cannot be greater than n, we see that it is in fact equal to n, i.e. (iii) holds.

Corollary 2.13

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Then the following are equivalent:

1. (i) $\mathbb{F}^n$ has a basis formed by eigenvectors of A;
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3. (iii) the sum of geometric multiplicities of all distinct eigenvalues A is equal to n.

Proof (continued). Reminder: WTS "(i) $\iff$ (iii)."

Suppose now that (iii) holds.

Corollary 2.13

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Then the following are equivalent:

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2. (ii) the sum of algebraic multiplicities of all distinct eigenvalues A is equal to n, and the geometric multiplicity of each eigenvalue of A is equal to the algebraic multiplicity of that eigenvalue;
3. (iii) the sum of geometric multiplicities of all distinct eigenvalues A is equal to n.

Proof (continued). Reminder: WTS "(i) $\iff$ (iii)."

Suppose now that (iii) holds. Let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues of A.

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Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Then the following are equivalent:

- (i) $\mathbb{F}^n$ has a basis formed by eigenvectors of A;
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- (iii) the sum of geometric multiplicities of all distinct eigenvalues A is equal to n.

Proof (continued). Reminder: WTS "(i) $\iff$ (iii)."

Suppose now that (iii) holds. Let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues of A. For each $i \in \{1, \dots, k\}$, let $\mathcal{B}_i$ be a basis for E_{λ_i};

Corollary 2.13

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3. (iii) the sum of geometric multiplicities of all distinct eigenvalues A is equal to n.

Proof (continued). Reminder: WTS "(i) $\iff$ (iii)."

Suppose now that (iii) holds. Let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues of A. For each $i \in \{1, \dots, k\}$, let $\mathcal{B}_i$ be a basis for E_{λ_i}; by (iii), we have that $|\mathcal{B}_1 \cup \dots \cup \mathcal{B}_k| = n$.

Corollary 2.13

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Then the following are equivalent:

1. (i) $\mathbb{F}^n$ has a basis formed by eigenvectors of A;
2. (ii) the sum of algebraic multiplicities of all distinct eigenvalues A is equal to n, and the geometric multiplicity of each eigenvalue of A is equal to the algebraic multiplicity of that eigenvalue;
3. (iii) the sum of geometric multiplicities of all distinct eigenvalues A is equal to n.

Proof (continued). Reminder: WTS "(i) $\iff$ (iii)."

Suppose now that (iii) holds. Let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues of A. For each $i \in \{1, \dots, k\}$, let $\mathcal{B}_i$ be a basis for E_{λ_i}; by (iii), we have that $|\mathcal{B}_1 \cup \dots \cup \mathcal{B}_k| = n$. On the other hand, by Theorem 2.12, $\mathcal{B}_1 \cup \dots \cup \mathcal{B}_k$ is a linearly independent set of vectors in $\mathbb{F}^n$.

Corollary 2.13

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Then the following are equivalent:

1. (i) $\mathbb{F}^n$ has a basis formed by eigenvectors of A;
2. (ii) the sum of algebraic multiplicities of all distinct eigenvalues A is equal to n, and the geometric multiplicity of each eigenvalue of A is equal to the algebraic multiplicity of that eigenvalue;
3. (iii) the sum of geometric multiplicities of all distinct eigenvalues A is equal to n.

Proof (continued). Reminder: WTS "(i) $\iff$ (iii)."

Suppose now that (iii) holds. Let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues of A. For each $i \in \{1, \dots, k\}$, let $\mathcal{B}_i$ be a basis for E_{λ_i}; by (iii), we have that $|\mathcal{B}_1 \cup \dots \cup \mathcal{B}_k| = n$. On the other hand, by Theorem 2.12, $\mathcal{B}_1 \cup \dots \cup \mathcal{B}_k$ is a linearly independent set of vectors in $\mathbb{F}^n$. So, by Proposition 1.11(a) of Lecture Notes 7, $\mathcal{B}_1 \cup \dots \cup \mathcal{B}_k$ is in fact a basis of $\mathbb{F}^n$, and so (i) holds. Q.E.D.