Linear Algebra 2

Lecture #17
Applications of determinants: polynomials and volume

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March 30, 2023

- This lecture has three parts:
  1. the Vandermonde matrix;
  2. common roots of polynomials via determinants;
  3. determinants and volume

1. The Vandermonde matrix

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Definition

For a positive integer n and real numbers $a_0, a_1, \ldots, a_n$, we define the matrix

$$V(a_0, a_1, \ldots, a_n) := \begin{bmatrix} 1 & 1 & \ldots & 1 \\ a_0 & a_1 & \ldots & a_n \\ a_0^2 & a_1^2 & \ldots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_0^n & a_1^n & \ldots & a_n^n \end{bmatrix}_{(n+1)\times(n+1)},$$

called the *Vandermonde matrix*.

Proposition 1.1

For all positive integers n and real numbers $a_0, a_1, \ldots, a_n$, we have that

$$\det\Big(V(a_0, a_1, \ldots, a_n)\Big) = \prod_{i>j}(a_i - a_j) = \prod_{i=1}^{n}\prod_{j=0}^{i-1}(a_i - a_j).$$

and consequently, the matrix $V(a_0, a_1, \ldots, a_n)$ is invertible iff $a_0, a_1, \ldots, a_n$ are pairwise distinct.

Proof.

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Proof. By Theorem 5.1 from Lecture Notes 15, we know that a square matrix is invertible iff its determinant is non-zero. So, the second statement of the proposition follows immediately from the first, i.e. from the formula for the determinant of the Vandermonde matrix.

Proof (continued). WTS $\det\Big(V(a_0, a_1, \ldots, a_n)\Big) = \prod_{i=1}^{n} \prod_{j=0}^{i-1} (a_i - a_j)$.

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$$\det\Big(V(a_0, a_1)\Big) = \begin{vmatrix} 1 & 1 \\ a_0 & a_1 \end{vmatrix} = (a_1 - a_0).$$

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Now, fix a positive integer n, and assume inductively that our formula is correct for n, i.e. that for all real numbers $a_0, a_1, \ldots, a_n$, we have that $\det\Big(V(a_0, a_1, \ldots, a_n)\Big) = \prod_{i=1}^{n} \prod_{j=0}^{i-1} (a_i - a_j)$. WTS the formula is correct for $n + 1$.

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If some two of $a_0, \dots, a_{n+1}$ are the same, then this is obvious. Indeed, in this case, the matrix $V(a_0, a_1, \dots, a_n, a_{n+1})$ has two identical columns and therefore has determinant zero, and clearly, $\prod_{i=1}^{n+1} \prod_{j=0}^{i} (a_i - a_j) = 0$ (because one of the factors is zero).

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Proof (continued). Reminder: $a_0, \ldots, a_{n+1}$ are pairwise distinct;

induction hypothesis: $\det\Big(V(a_0, a_1, \ldots, a_n)\Big) = \prod_{i=1}^{n} \prod_{j=0}^{i-1} (a_i - a_j)$;

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$$f(t) \quad := \quad \begin{vmatrix} 1 & 1 & \ldots & 1 & 1 \\ a_0 & a_1 & \ldots & a_n & t \\ a_0^2 & a_1^2 & \ldots & a_n^2 & t^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_0^n & a_1^n & \ldots & a_n^n & t^n \\ a_0^{n+1} & a_1^{n+1} & \ldots & a_n^{n+1} & t^{n+1} \end{vmatrix},$$

so that $f(a_{n+1}) = V(a_0, a_1, \ldots, a_n, a_{n+1})$.

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so that $f(a_{n+1}) = V(a_0, a_1, \ldots, a_n, a_{n+1})$. By performing Laplace expansion along the last column, we see that $f(t)$ is a polynomial of degree $n+1$, and that its leading coefficient (i.e. coefficient in front of t^{n+1}) is $\det\Big(V(a_0, a_1, \ldots, a_n)\Big) \overset{\text{ind. hyp.}}{=} \prod_{i=1}^{n} \prod_{j=0}^{i-1} (a_i - a_j)$.

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so that $f(a_{n+1}) = V(a_0, a_1, \dots, a_n, a_{n+1})$. By performing Laplace expansion along the last column, we see that $f(t)$ is a polynomial of degree $n+1$, and that its leading coefficient (i.e. coefficient in front of t^{n+1}) is $\det\Big(V(a_0, a_1, \dots, a_n)\Big) \overset{\text{ind. hyp.}}{=} \prod_{i=1}^{n} \prod_{j=0}^{i-1} (a_i - a_j)$. Also, $a_0, \dots, a_n$ are the roots of $f(t)$.

Proof (continued). Reminder: $a_0, \dots, a_{n+1}$ are pairwise distinct; induction hypothesis: $\det\Big(V(a_0, a_1, \dots, a_n)\Big) = \prod_{i=1}^{n} \prod_{j=0}^{i-1} (a_i - a_j)$;

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Proof (continued). Reminder: $a_0, \ldots, a_{n+1}$ are pairwise distinct; induction hypothesis: $\det\Big(V(a_0, a_1, \ldots, a_n)\Big) = \prod_{i=1}^{n} \prod_{j=0}^{i-1} (a_i - a_j)$;
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$$f(t) \;=\; k \prod_{j=0}^{n} (t - a_j) \;=\; \Big(\prod_{i=1}^{n} \prod_{j=0}^{i-1} (a_i - a_j) \Big) \Big(\prod_{j=0}^{n} (t - a_j) \Big).$$

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Consequently,

$$\det\Big(V(a_0, a_1, \ldots, a_n, a_{n+1})\Big) \;=\; f(a_{n+1}) \;=\; \prod_{i=1}^{n+1} \prod_{j=0}^{i-1} (a_i - a_j).$$

This completes the induction. Q.E.D.

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  - Examples: Lecture Notes.

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- More precisely, for such a polynomial $p(x)$, there exist complex numbers $a, \alpha_1, \dots, \alpha_\ell$ s.t. $a \neq 0$ and s.t. $\alpha_1, \dots, \alpha_\ell$ are pairwise distinct, and positive integers $n_1, \dots, n_\ell$ satisfying $n_1 + \cdots + n_\ell = n$, s.t.

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$$p(x) = a(x - \alpha_1)^{n_1} \dots (x - \alpha_\ell)^{n_\ell}.$$

- Here, a is the leading coefficient of $p(x)$, i.e. the coefficient in front of x^n. Complex numbers $\alpha_1, \dots, \alpha_\ell$ are the roots of $p(x)$ with *multiplicities* $n_1, \dots, n_\ell$, respectively.

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- In fact, not only is no such formula known, but using Galois theory, one can show that no such formula can exist for polynomials of degree at least five.

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- In fact, not only is no such formula known, but using Galois theory, one can show that no such formula can exist for polynomials of degree at least five.
    - Of course, we may be able to use various tricks to compute the roots of some special high-degree polynomials. However, none of these tricks will work in the general case.

Theorem 2.1

Let m and n be positive integers, and let $p(x) = \sum_{i=0}^{m} a_i x^i$ (with $a_m \neq 0$) and $q(x) = \sum_{i=0}^{n} b_i x^i$ ($b_n \neq 0$) be polynomials with complex coefficients. Let P be the $n \times (n+m)$ matrix whose j-th row (for $j \in \{1, \ldots, n\}$) is

$$\left[\quad \underbrace{0 \quad \ldots \quad 0}_{j-1} \quad a_m \quad a_{m-1} \quad \ldots \quad a_0 \quad \underbrace{0 \quad \ldots \quad 0}_{n-j} \quad \right],$$

and let Q be the $m \times (n+m)$ matrix whose j-th row (for $j \in \{1, \ldots, m\}$) is

$$\left[\quad \underbrace{0 \quad \ldots \quad 0}_{j-1} \quad b_n \quad b_{n-1} \quad \ldots \quad b_0 \quad \underbrace{0 \quad \ldots \quad 0}_{m-j} \quad \right].$$

Then $p(x)$ and $q(x)$ have a common complex root iff

$$\det\left(\left[\begin{array}{c} P \\ \hline Q \end{array} \right] \right) \quad = \quad 0.$$

- **Remark#1:** For example, if $m = 3$ and $n = 5$, so that
  - $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$,
  - $q(x) = b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0$,

  then we have

$$\left[\begin{array}{c} P \\ \hdashline Q \end{array}\right] = \left[\begin{array}{cccccccc} a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 \\ 0 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & 0 & 0 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 0 & 0 & a_3 & a_2 & a_1 & a_0 \\ \hdashline b_5 & b_4 & b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & b_5 & b_4 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & b_5 & b_4 & b_3 & b_2 & b_1 & b_0 \end{array}\right]_{8\times 8}.$$

- **Remark#2:** Theorem 2.1 works for polynomials with **complex** coefficients, but not for polynomials with coefficients in an arbitrary field $\mathbb{F}$. This is essentially because the proof of Theorem 2.1 relies on the Fundamental Theorem of Algebra, which works for $\mathbb{C}$, but not for general fields $\mathbb{F}$.

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- **Remark#3:** Theorem 2.1 gives a recipe for determining whether two polynomials with complex coefficients have a complex root, but it does not give a recipe for actually computing such a common root (if it exists).

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Let m and n be positive integers, and let $p(x) = \sum_{i=0}^{m} a_i x^i$ (with $a_m \neq 0$) and $q(x) = \sum_{i=0}^{n} b_i x^i$ ($b_n \neq 0$) be polynomials with complex coefficients. Let P be the $n \times (n+m)$ matrix whose j-th row (for $j \in \{1, \ldots, n\}$) is

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Then $p(x)$ and $q(x)$ have a common complex root iff

$$\det\left(\left[\frac{P}{Q} \right] \right) = 0.$$

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Example 2.2

Determine whether the polynomials $p(x) = 5x^3 - 2x^2 + x - 4$ and $q(x) = 7x^2 - 6x - 1$ have a common complex root.

Solution.

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Example 2.2

Determine whether the polynomials $p(x) = 5x^3 - 2x^2 + x - 4$ and $q(x) = 7x^2 - 6x - 1$ have a common complex root.

Solution. In this case, it is easy to see that $p(1) = 0$ and $q(1) = 0$, and so 1 is a common root of $p(x)$ and $q(x)$. However, let us use Theorem 2.1, in order to illustrate how this theorem can be applied.

Using the notation of Theorem 2.1, we have that $m = 3$, $n = 2$, and the matrices P and Q are given by

- $P = \begin{bmatrix} 5 & -2 & 1 & -4 & 0 \\ 0 & 5 & -2 & 1 & -4 \end{bmatrix}$;
- $Q = \begin{bmatrix} 7 & -6 & -1 & 0 & 0 \\ 0 & 7 & -6 & -1 & 0 \\ 0 & 0 & 7 & -6 & -1 \end{bmatrix}$.

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Solution (continued). We now have that

$$\det\left(\left[\begin{array}{c} P \\ \hline Q \end{array}\right]\right) = \left|\begin{array}{ccccc} 5 & -2 & 1 & -4 & 0 \\ 0 & 5 & -2 & 1 & -4 \\ \hline 7 & -6 & -1 & 0 & 0 \\ 0 & 7 & -6 & -1 & 0 \\ 0 & 0 & 7 & -6 & -1 \end{array}\right| = 0.$$

Theorem 2.1 now guarantees that $p(x)$ and $q(x)$ have a common complex root.

Theorem 2.1

Let m and n be positive integers, and let $p(x) = \sum_{i=0}^{m} a_i x^i$ (with $a_m \neq 0$) and $q(x) = \sum_{i=0}^{n} b_i x^i$ ($b_n \neq 0$) be polynomials with complex coefficients. Let P be the $n \times (n+m)$ matrix whose j-th row (for $j \in \{1, \ldots, n\}$) is

$$\left[\quad \underbrace{0 \quad \ldots \quad 0}_{j-1} \quad a_m \quad a_{m-1} \quad \ldots \quad a_0 \quad \underbrace{0 \quad \ldots \quad 0}_{n-j} \quad \right],$$

and let Q be the $m \times (n+m)$ matrix whose j-th row (for $j \in \{1, \ldots, m\}$) is

$$\left[\quad \underbrace{0 \quad \ldots \quad 0}_{j-1} \quad b_n \quad b_{n-1} \quad \ldots \quad b_0 \quad \underbrace{0 \quad \ldots \quad 0}_{m-j} \quad \right].$$

Then $p(x)$ and $q(x)$ have a common complex root iff

$$\det\left(\left[\frac{P}{Q} \right] \right) = 0.$$

Proof.

Claim. *Polynomials $p(x)$ and $q(x)$ have a common complex root iff there exist non-zero polynomials $r(x)$ and $s(x)$ with complex coefficients that satisfy the following:*

- *$deg(r(x)) \leq n-1$;*
- *$deg(s(x)) \leq m-1$;*
- *$r(x)p(x) + s(x)q(x) = 0$.*

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Proof of the Claim. Suppose first that $p(x)$ and $q(x)$ have a common root, say α. Then we set $r(x) := \frac{q(x)}{x-\alpha}$ and $s(x) := -\frac{p(x)}{x-\alpha}$, and we observe that $\deg(r(x)) = \deg(q(x)) - 1 = n - 1$, $\deg(s(x)) = \deg(p(x)) - 1 = m - 1$, and

$$r(x)p(x) + s(x)q(x) \quad = \quad \tfrac{q(x)p(x)}{x-\alpha} - \tfrac{p(x)q(x)}{x-\alpha} \quad = \quad 0.$$

Proof (continued).

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Proof of the Claim (continued). Suppose conversely there exist $r(x), s(x)$ with the desired properties. Then $r(x)p(x)$ and $s(x)q(x)$ are non-constant polynomials with complex coefficients, and they have exactly the same roots with the same corresponding multiplicities.

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Proof of the Claim (continued). Suppose conversely there exist $r(x), s(x)$ with the desired properties. Then $r(x)p(x)$ and $s(x)q(x)$ are non-constant polynomials with complex coefficients, and they have exactly the same roots with the same corresponding multiplicities.

Since $\deg(p(x)) = m$, the Fundamental Theorem of Algebra implies that $p(x)$ has exactly m complex roots (when multiplicities are taken into account). But $\deg(s(x)) \leq m-1$, and so at least one of the roots of $p(x)$ either fails to be a root of $s(x)$, or is a root of $s(x)$ but has smaller multiplicity in $s(x)$ than in $p(x)$. This root of $p(x)$ must therefore be a root of $q(x)$. This proves the Claim.

Proof (continued). In view of the Claim, it now suffices to determine if there exist non-zero polynomials $r(x) = \sum_{i=0}^{n-1} c_i x^i$ and $s(x) = \sum_{i=0}^{m-1} d_i x^i$ s.t. $r(x)p(x) + s(x)q(x) = 0$. So, we need to determine if there exist complex numbers $c_0, \dots, c_{n-1}, d_0, \dots, d_{m-1}$ s.t. at least one of $c_0, \dots, c_{n-1}$ is non-zero and at least one of $d_0, \dots, d_{m-1}$ is non-zero, and s.t.

$$\underbrace{\Big(\sum_{i=0}^{n-1} c_i x^i\Big)}_{=r(x)}\underbrace{\Big(\sum_{i=0}^{m} a_i x^i\Big)}_{=p(x)} + \underbrace{\Big(\sum_{i=0}^{m-1} d_i x^i\Big)}_{=s(x)}\underbrace{\Big(\sum_{i=0}^{n} b_i x^i\Big)}_{=q(x)} \quad = \quad 0.$$

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$$\underbrace{\Big(\sum_{i=0}^{n-1} c_i x^i\Big)}_{=r(x)} \underbrace{\Big(\sum_{i=0}^{m} a_i x^i\Big)}_{=p(x)} + \underbrace{\Big(\sum_{i=0}^{m-1} d_i x^i\Big)}_{=s(x)} \underbrace{\Big(\sum_{i=0}^{n} b_i x^i\Big)}_{=q(x)} = 0.$$

But obviously, if $c_0, \dots, c_{n-1}$ are all zero, then $d_0, \dots, d_{m-1}$ are all zero, and vice versa. So, we in fact need to determine if the above equality holds for some complex numbers $c_0, \dots, c_{n-1}, d_0, \dots, d_{m-1}$, at least one of which is non-zero.

Proof (continued). Reminder:

$$\underbrace{\left(\sum_{i=0}^{n-1} c_i x^i\right)}_{=r(x)} \underbrace{\left(\sum_{i=0}^{m} a_i x^i\right)}_{=p(x)} + \underbrace{\left(\sum_{i=0}^{m-1} d_i x^i\right)}_{=s(x)} \underbrace{\left(\sum_{i=0}^{n} b_i x^i\right)}_{=q(x)} \quad = \quad 0.$$

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We now write the polynomial on the left-hand-side in the standard form, and we set all the coefficients that we obtain equal to zero.

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Proof (continued). Reminder:

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$$A \left[\begin{array}{ccc:ccc} c_{n-1} & \dots & c_0 & d_{m-1} & \dots & d_0 \end{array}\right]^T \quad = \quad \mathbf{0},$$

and we observe that the coefficient matrix A satisfies $A^T = \left[\begin{array}{c} P \\ \hdashline Q \end{array}\right]$.

Proof (continued). We now have the following equivalences:

$$
\begin{array}{lll}
\begin{array}{l} p(x) \text{ and } q(x) \\ \text{have a common} \\ \text{complex root} \end{array} & \iff & \begin{array}{l} A \left[\begin{array}{ccc:ccc} c_{n-1} & \dots & c_0 & d_{m-1} & \dots & d_0 \end{array} \right]^T = \mathbf{0} \\ \text{has a non-zero solution} \end{array} \\
& \iff & A \text{ is non-invertible} \\
& \iff & A^T = \left[\begin{array}{c} P \\ \hline Q \end{array} \right] \text{ is non-invertible} \\
& \iff & \det\left(\left[\begin{array}{c} P \\ \hline Q \end{array} \right] \right) = 0,
\end{array}
$$

Q.E.D.

3. Determinants and volume

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- From now on, we assume that $\mathbb{R}^n$ is equipped with the standard scalar product $\cdot$ and the induced norm $||\cdot||$.

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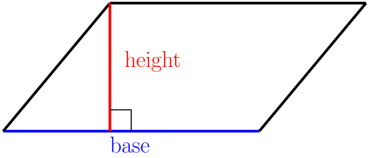
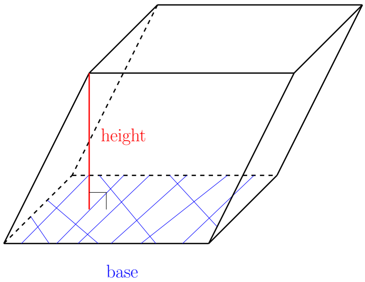
- For a parallelogram, we have the familiar formula

$$\left(\begin{array}{c}\text{area of}\\\text{parallelogram}\end{array}\right) = (\text{length of base}) \times (\text{height}).$$



- We have a similar formula for the volume of a parallelepiped:

$$\begin{pmatrix} \text{volume of} \\ \text{parallelepiped} \end{pmatrix} = (\text{area of base}) \times (\text{height}).$$



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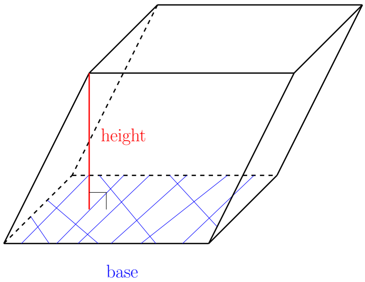

- We would now like to generalize this to arbitrary dimensions.

- Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$, the *m-parallelepiped* determined by vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is the set

$$\Big\{ c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m \in \mathbb{R}^n \;\mid\; c_1, \ldots, c_m \in \mathbb{R}, 0 \leq c_1, \ldots, c_m \leq 1 \Big\}.$$

- Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$, the *m-parallelepiped* determined by vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is the set

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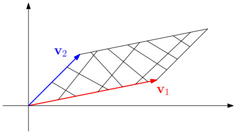
- For instance, given two vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$, neither of which is a scalar multiple of the other, the 2-parallelepiped determined by $\mathbf{v}_1, \mathbf{v}_2$ is a just a parallelogram.



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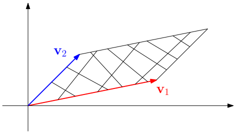

- For vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$, neither of which is a scalar multiple of each other, the 2-parallelepiped determined by $\mathbf{v}_1, \mathbf{v}_2$ is still a parallelogram, but this parallelogram lies in the plane (2-dimensional subspace) $\text{Span}(\mathbf{v}_1, \mathbf{v}_2)$ of $\mathbb{R}^n$.

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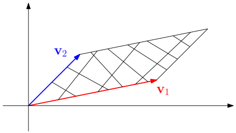

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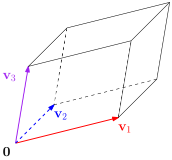
- If one of $\mathbf{v}_1, \mathbf{v}_2$ is a scalar multiple of the other, then we get a "degenerate parallelogram" (line segment or just $\{\mathbf{0}\}$).

- Similarly, for three linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^n$, the 3-parallelepiped defined by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is just the usual parallelepiped whose edges are determined by these three vectors.



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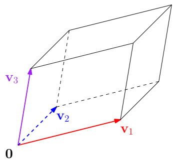

- If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not linearly independent, then the 3-parallelepiped determined by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is either a parallelogram, or a line segment, or $\{\mathbf{0}\}$, depending on the dimension of $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Once again, we can think of these as "degenerate parallelepipeds."

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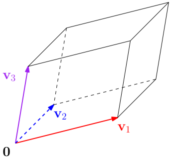

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- For more than three vectors, we get higher-dimensional generalizations.

- We would now like to define the "volume" (more precisely, the "m-volume") of an m-parallelepiped in $\mathbb{R}^n$.

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  - The *1-volume* of the 1-parallelepiped determined by the vector $\mathbf{v}_1 \in \mathbb{R}^n$ is defined to be

$$V_1(\mathbf{v}_1) \quad := \quad ||\mathbf{v}_1||.$$

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  - For a positive integer m, the $(m+1)$-volume of the $(m+1)$-parallelepiped determined by the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{v}_{m+1} \in \mathbb{R}^n$ is defined to be

$$V_{m+1}(\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{v}_{m+1}) \quad := \quad V_m(\mathbf{v}_1, \ldots, \mathbf{v}_m)\, ||\mathbf{v}_{m+1}^{\perp}||,$$

    where $\mathbf{v}_{m+1}^{\perp} = \mathrm{proj}_{\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)^{\perp}}(\mathbf{v}_{m+1})$, or equivalently, $\mathbf{v}_{m+1}^{\perp} = \mathbf{v}_{m+1} - \mathrm{proj}_{\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)}(\mathbf{v}_{m+1})$.

- Reminder:
  - $V_1(\mathbf{v}_1) := ||\mathbf{v}_1||$;
  - $V_{m+1}(\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{v}_{m+1}) := V_m(\mathbf{v}_1, \ldots, \mathbf{v}_m)\ ||\mathbf{v}_{m+1}^{\perp}||$.

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  - $V_{m+1}(\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{v}_{m+1}) := V_m(\mathbf{v}_1, \ldots, \mathbf{v}_m) \, ||\mathbf{v}_{m+1}^{\perp}||$.

- In this recursive formula, the m-parallelepiped determined by the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is our "base" and $||\mathbf{v}_{m+1}^{\perp}||$ is our "height."

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- Note that 1-volume represents (1-dimensional) length, 2-volume represents (2-dimensional) area, and 3-volume represents (3-dimensional) volume. For $m \geq 4$, m-volume is an m-dimensional generalization of these concepts.

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Proposition 3.1

Let $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$. Then $V_m(\mathbf{v}_1, \ldots, \mathbf{v}_m) \geq 0$, and equality holds iff $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is a linearly dependent set.

Corollary 3.4

Let $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n$ and $\sigma \in S_m$. Then $V_m(\mathbf{a}_1, \ldots, \mathbf{a}_m) = V_m(\mathbf{a}_{\sigma(1)}, \ldots, \mathbf{a}_{\sigma(m)})$.

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- The proof of Corollary 3.4 uses Theorem 3.2, which gives a formula for the volume in terms of the determinant of a certain square matrix (later!).

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Proof (outline). For each $i \in \{1, \ldots, m-1\}$, set $\mathbf{v}_{i+1}^{\perp} := \text{proj}_{\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_i)^{\perp}}(\mathbf{v}_{i+1})$.

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Since the length of any vector is non-negative, we see that $V_m(\mathbf{v}_1, \ldots, \mathbf{v}_m) \geq 0$. Moreover, equality holds iff at least one of the vectors $\mathbf{v}_1, \mathbf{v}_2^{\perp}, \ldots, \mathbf{v}_m^{\perp}$ is $\mathbf{0}$.

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Since the length of any vector is non-negative, we see that $V_m(\mathbf{v}_1, \ldots, \mathbf{v}_m) \geq 0$. Moreover, equality holds iff at least one of the vectors $\mathbf{v}_1, \mathbf{v}_2^{\perp}, \ldots, \mathbf{v}_m^{\perp}$ is $\mathbf{0}$. But the latter happens iff the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is linearly dependent (details: Lecture Notes).

Theorem 3.2

Let $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n$, and set $A := \begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_m \end{bmatrix}$. Then

$$V_m(\mathbf{a}_1, \ldots, \mathbf{a}_m) = \sqrt{\det(A^T A)}.$$

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For $i = 1$, we observe that

$$A_1^T A_1 \quad = \quad \begin{bmatrix} \mathbf{a}_1 \end{bmatrix}^T \begin{bmatrix} \mathbf{a}_1 \end{bmatrix} \quad = \quad \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{a}_1 \end{bmatrix},$$

and consequently,

$$\sqrt{\det(A_1^T A_1)} \quad = \quad \sqrt{\mathbf{a}_1 \cdot \mathbf{a}_1} \quad = \quad ||\mathbf{a}_1|| \quad = \quad V_1(\mathbf{a}_1).$$

Proof (continued). We may now assume that $m \geq 2$, for otherwise we are done by what we just showed.

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$$\mathbf{a}_{i+1}^{\perp} \;=\; \mathbf{a}_{i+1} - \mathbf{a}_i^{||} \;=\; \mathbf{a}_{i+1} - c_1\mathbf{a}_1 - \dots - c_i\mathbf{a}_i.$$

Proof (continued). We may now assume that $m \geq 2$, for otherwise we are done by what we just showed. Fix $i \in \{1, \ldots, m-1\}$, and assume inductively that $V_i(\mathbf{a}_1, \ldots, \mathbf{a}_i) = \sqrt{\det(A_i^T A_i)}$. WTS $V_{i+1}(\mathbf{a}_1, \ldots, \mathbf{a}_i, \mathbf{a}_{i+1}) = \sqrt{\det(A_{i+1}^T A_{i+1})}$. Set

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Now, let B_{i+1} be the matrix obtained from A_{i+1} by replacing the last column of A_{i+1} by $\mathbf{a}_{i+1}^{\perp}$, i.e.

$$B_{i+1} \;\; := \;\; \begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_i & \mathbf{a}_{i+1}^{\perp} \end{bmatrix}.$$

Proof (continued). Then

$$B_{i+1}^T = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_i^T \\ (\mathbf{a}_{i+1}^\perp)^T \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_i^T \\ \mathbf{a}_{i+1}^T - c_1\mathbf{a}_1^T - \cdots - c_i\mathbf{a}_i^T \end{bmatrix}.$$

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So, B_{i+1}^T can be obtained from A_{i+1}^T via the following sequence of i elementary row operations:

- $R_{i+1} \to R_{i+1} - c_1 R_1$;

  $\vdots$

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Proof (continued). Then

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Let $E_1, \dots, E_i$ be the elementary matrices corresponding to these i elementary row operations, so that $B_{i+1}^T = E_i \dots E_1 A_{i+1}^T$, and consequently, $B_{i+1} = A_{i+1} E_1^T \dots E_i^T$.

Proof (continued). Then

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Proof (continued). We now compute:

$$\begin{aligned}
\det(B_{i+1}^T B_{i+1}) &= \det\Big((E_i \dots E_1 A_{i+1}^T)(A_{i+1} E_1^T \dots E_i^T)\Big) \\
&= \det(E_i) \dots \det(E_1) \det(A_{i+1}^T A_{i+1}) \det(E_1^T) \dots \det(E_i^T) \\
&= \underbrace{\det(E_i)}_{=1} \dots \underbrace{\det(E_1)}_{=1} \det(A_{i+1}^T A_{i+1}) \underbrace{\det(E_1)}_{=1} \dots \underbrace{\det(E_i)}_{=1} \\
&= \det(A_{i+1}^T A_{i+1}),
\end{aligned}$$

Proof (continued). But note that $B_{i+1} = \left[\begin{array}{c:c} A_i & \mathbf{a}_{i+1}^{\perp} \end{array}\right]$, and so

$$\begin{aligned}
B_{i+1}^T B_{i+1} &= \left[\begin{array}{c} A_i^T \\ \hdashline (\mathbf{a}_{i+1}^{\perp})^T \end{array}\right] \left[\begin{array}{c:c} A_i & \mathbf{a}_{i+1}^{\perp} \end{array}\right] \\
&= \left[\begin{array}{c:c} A_i^T A_i & A_i^T \mathbf{a}_{i+1}^{\perp} \\ \hdashline (\mathbf{a}_{i+1}^{\perp})^T A_i & (\mathbf{a}_{i+1}^{\perp})^T \mathbf{a}_{i+1}^{\perp} \end{array}\right] \\
&= \left[\begin{array}{c:c} A_i^T A_i & A_i^T \mathbf{a}_{i+1}^{\perp} \\ \hdashline (A_i^T \mathbf{a}_{i+1}^{\perp})^T & \mathbf{a}_{i+1}^{\perp} \cdot \mathbf{a}_{i+1}^{\perp} \end{array}\right] \\
&\overset{(*)}{=} \left[\begin{array}{c:c} A_i^T A_i & \mathbf{0} \\ \hdashline \mathbf{0}^T & ||\mathbf{a}_{i+1}^{\perp}||^2 \end{array}\right],
\end{aligned}$$

where in (*), we used the fact that $\mathbf{a}_{i+1}^{\perp}$ is orthogonal to the columns of A, and so $A^T \mathbf{a}_{i+1}^{\perp} = \mathbf{0}$.

Proof (continued). We now compute:

$$\begin{aligned}
\det(A_{i+1}^T A_{i+1}) &= \det(B_{i+1}^T B_{i+1}) \\
&= \left| \begin{array}{c:c} A_i^T A_i & \mathbf{0} \\ \hdashline \mathbf{0}^T & ||\mathbf{a}_{i+1}^{\perp}||^2 \end{array} \right| \\
&\overset{(*)}{=} (-1)^{(i+1)+(i+1)} \, ||\mathbf{a}_{i+1}^{\perp}||^2 \det(A_i^T A_i) \\
&= \det(A_i^T A_i) \, ||\mathbf{a}_{i+1}^{\perp}||^2 \\
&\overset{(**)}{=} V_i(\mathbf{a}_1, \ldots, \mathbf{a}_i)^2 \, ||\mathbf{a}_{i+1}^{\perp}||^2 \\
&\overset{(***)}{=} V_{i+1}(\mathbf{a}_1, \ldots, \mathbf{a}_i, \mathbf{a}_{i+1})^2,
\end{aligned}$$

where (*) follows by Laplace expansion along the last column, (**) follows from the induction hypothesis, and (***) follows from the definition of $V_{i+1}(\mathbf{a}_1, \ldots, \mathbf{a}_i, \mathbf{a}_{i+1})$.

Proof (continued). Reminder:
$\det(A_{i+1}^T A_{i+1}) = V_{i+1}(\mathbf{a}_1, \ldots, \mathbf{a}_i, \mathbf{a}_{i+1})^2$.

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Since $V_{i+1}(\mathbf{a}_1, \ldots, \mathbf{a}_i, \mathbf{a}_{i+1}) \geq 0$ (by Proposition 3.1), we may now take the square root of both sides to obtain

$$V_{i+1}(\mathbf{a}_1, \ldots, \mathbf{a}_i, \mathbf{a}_{i+1}) \quad = \quad \sqrt{\det(A_{i+1}^T A_{i+1})}.$$

This completes the induction. Q.E.D.

Theorem 3.2

Let $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n$, and set $A := \begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_m \end{bmatrix}$. Then

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Corollary 3.3

Let $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^n$. Then

$$V_n(\mathbf{a}_1, \ldots, \mathbf{a}_n) = |\det(\begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_n \end{bmatrix})|.$$

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Proof.

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Proof. First of all, we note that $A := \begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_n \end{bmatrix}$ is an $n \times n$ matrix (with entries in $\mathbb{R}$), and so it has a determinant.

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$$\begin{aligned}
V_n(\mathbf{a}_1, \ldots, \mathbf{a}_n) &= \sqrt{\det(A^T A)} && \text{by Theorem 3.2} \\
&= \sqrt{\det(A^T)\det(A)} \\
&= \sqrt{\det(A)^2} \\
&= |\det(A)|.
\end{aligned}$$

Q.E.D.

Corollary 3.4

Let $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n$ and $\sigma \in S_m$. Then
$V_m(\mathbf{a}_1, \ldots, \mathbf{a}_m) = V_m(\mathbf{a}_{\sigma(1)}, \ldots, \mathbf{a}_{\sigma(m)})$.

Proof.

Corollary 3.4

Let $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n$ and $\sigma \in S_m$. Then $V_m(\mathbf{a}_1, \ldots, \mathbf{a}_m) = V_m(\mathbf{a}_{\sigma(1)}, \ldots, \mathbf{a}_{\sigma(m)})$.

Proof. Set $A := \begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_m \end{bmatrix}$ and $A_\sigma := \begin{bmatrix} \mathbf{a}_{\sigma(1)} & \ldots & \mathbf{a}_{\sigma(m)} \end{bmatrix}$. Further, let $E_\sigma = [e_{i,j}]_{m\times m}$ be the matrix given by

$$e_{i,j} = \begin{cases} 1 & \text{if} \quad j = \sigma(i) \\ 0 & \text{if} \quad j \neq \sigma(i) \end{cases}$$

for all $i, j \in \{1, \ldots, m\}$.

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Proof. Set $A := \begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_m \end{bmatrix}$ and $A_\sigma := \begin{bmatrix} \mathbf{a}_{\sigma(1)} & \ldots & \mathbf{a}_{\sigma(m)} \end{bmatrix}$. Further, let $E_\sigma = [e_{i,j}]_{m \times m}$ be the matrix given by

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for all $i, j \in \{1, \ldots, m\}$. Note that $\det(E_\sigma) = \operatorname{sgn}(\sigma)$.

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Proof. Set $A := \begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_m \end{bmatrix}$ and $A_\sigma := \begin{bmatrix} \mathbf{a}_{\sigma(1)} & \ldots & \mathbf{a}_{\sigma(m)} \end{bmatrix}$. Further, let $E_\sigma = [e_{i,j}]_{m \times m}$ be the matrix given by

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for all $i, j \in \{1, \ldots, m\}$. Note that $\det(E_\sigma) = \text{sgn}(\sigma)$. Moreover, $A_\sigma^T = E_\sigma A^T$, and so $A_\sigma = A E_\sigma^T$.

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Proof (continued). But now

$$\begin{aligned}
V_m(\mathbf{a}_{\sigma(1)}, \ldots, \mathbf{a}_{\sigma(m)}) &\overset{\text{Theorem 3.2}}{=} \sqrt{\det(A_\sigma^T A_\sigma)} \\
&= \sqrt{\det(E_\sigma A^T A E_\sigma^T)} \\
&= \sqrt{\det(E_\sigma)\det(A^T A)\det(E_\sigma^T)} \\
&= \sqrt{\det(E_\sigma)\det(A^T A)\det(E_\sigma)} \\
&= \sqrt{\operatorname{sgn}(\sigma)^2 \det(A^T A)} \\
&= \sqrt{\det(A^T A)} \\
&\overset{\text{Theorem 3.2}}{=} V_m(\mathbf{a}_1, \ldots, \mathbf{a}_m)
\end{aligned}$$

Q.E.D.

Corollary 3.5

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^n$, and let $A \in \mathbb{R}^{n \times n}$. Then

$$V_n(A\mathbf{v}_1, \ldots, A\mathbf{v}_n) \quad = \quad |\det(A)| \; V_n(\mathbf{v}_1, \ldots, \mathbf{v}_n).$$

Proof.

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$$V_n(A\mathbf{v}_1, \ldots, A\mathbf{v}_n) \quad = \quad |\det(A)| \; V_n(\mathbf{v}_1, \ldots, \mathbf{v}_n).$$

Proof. Set $B := \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{bmatrix}$ and $C := \begin{bmatrix} A\mathbf{v}_1 & \ldots & A\mathbf{v}_n \end{bmatrix} = AB$.

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$$V_n(A\mathbf{v}_1, \ldots, A\mathbf{v}_n) \quad = \quad |\det(A)|\ V_n(\mathbf{v}_1, \ldots, \mathbf{v}_n).$$

Proof. Set $B := [\ \mathbf{v}_1 \quad \ldots \quad \mathbf{v}_n\ ]$ and $C := [\ A\mathbf{v}_1 \quad \ldots \quad A\mathbf{v}_n\ ] = AB$. Note that A, B, C, AB all belong to $\mathbb{R}^{n \times n}$, and so all four matrices have determinants. We now compute:

$$\begin{aligned}
V(A\mathbf{v}_1, \ldots, A\mathbf{v}_n) &\overset{\text{Theorem 3.2}}{=} \sqrt{\det(C^T C)} \\
&= \sqrt{\det\Big((AB)^T (AB)\Big)} \\
&= \sqrt{\det(B^T A^T AB)} \\
&= \sqrt{\det(B^T)\det(A^T)\det(A)\det(B)} \\
&= \sqrt{\det(A)^2 \det(B^T)\det(B)} \\
&= \sqrt{\det(A)^2\ \det(B^T B)} \\
&= |\det(A)| \sqrt{\det(B^T B)} \\
&\overset{\text{Theorem 3.2}}{=} |\det(A)|\ V_m(\mathbf{v}_1, \ldots, \mathbf{v}_n)
\end{aligned}$$

Q.E.D.

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Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^n$, and let $A \in \mathbb{R}^{n \times n}$. Then

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- **Remark:** For $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n$ $(m \neq n)$ and $A \in \mathbb{R}^{n \times n}$, the formula from Corollary 3.5 fails, i.e.

$$V_m(A\mathbf{v}_1, \ldots, A\mathbf{v}_m) \;\neq\; |\det(A)|\; V_m(\mathbf{v}_1, \ldots, \mathbf{v}_m).$$

  - For instance, for $m = 1$ and $n = 2$, we can take $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, so that $A\mathbf{v}_1 = \mathbf{v}_1$.

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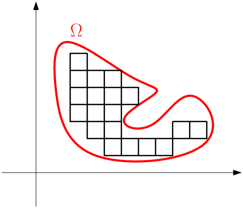
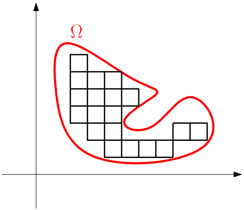
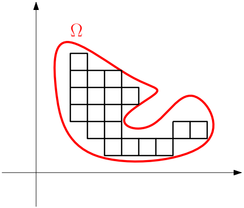
  - Then $V_1(\mathbf{v}_1) = V_1(A\mathbf{v}_1) = ||\mathbf{v}_1|| = 1$, but $\det(A) = 0$, and so $V_1(A\mathbf{v}_1) \neq |\det(A)| \; V_1(\mathbf{v}_1)$.



- Suppose that Ω is any object in $\mathbb{R}^n$ for which n-volume $V_n(\Omega)$ can be defined.



- Suppose that Ω is any object in $\mathbb{R}^n$ for which n-volume $V_n(\Omega)$ can be defined.
    - The idea is that we approximate Ω with ever smaller n-dimensional hypercubes; the sum of n-volumes of those n-hypercubes will give us an ever better approximation of the n-volume of Ω that we wish to define.



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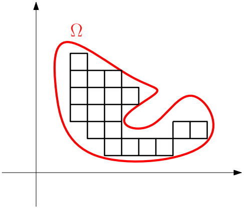
  - To obtain the actual n-volume of Ω, we take the limit of these ever-finer approximations. If the limit exists, then Ω will have an n-volume (defined to be this limit). If the limit does not exist, then n-volume is undefined for Ω.



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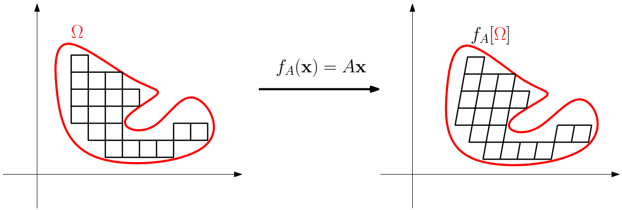
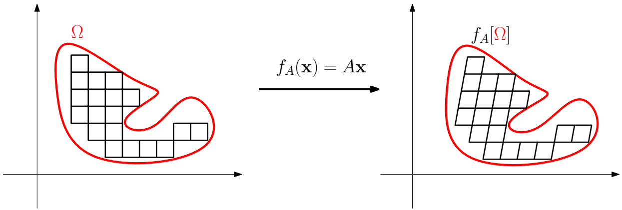
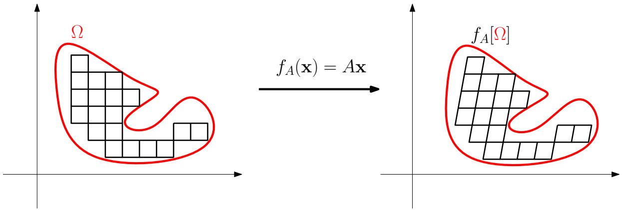
  - Details: omitted!



- Now, suppose we are given a matrix $A \in \mathbb{R}^{n \times n}$.



- Now, suppose we are given a matrix $A \in \mathbb{R}^{n \times n}$.
- We consider the linear function $f_A : \mathbb{R}^n \to \mathbb{R}^n$ whose standard matrix is A (i.e. for all $\mathbf{x} \in \mathbb{R}^n$, we have $f_A(\mathbf{x}) = A\mathbf{x}$).



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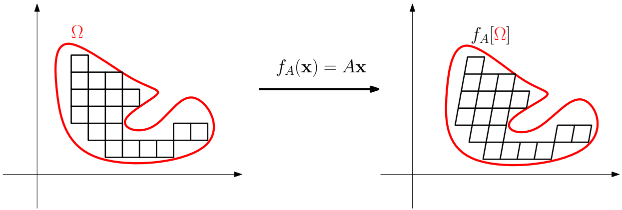
- Then each of the small n-hypercubes gets mapped onto a small n-parallelepiped; if the small n-hypercubes each had volume V, then by Corollary 3.5, the small n-parallelepipeds that these n-hypercubes get mapped onto via f_A will have volume $|\det(A)|\, V$.



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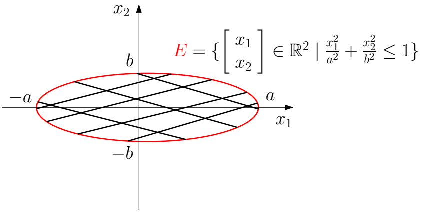
- So, we get the following formula for the n-volume of the image of Ω under f_A:

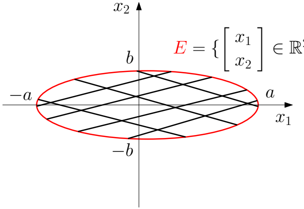
$$V_n(f_A[\Omega]) \quad = \quad |\det(A)|\ V_n(\Omega).$$

Example 3.6

Let a and b be positive real numbers. Compute the area (i.e. 2-volume) of the region bounded by the ellipse whose equation is

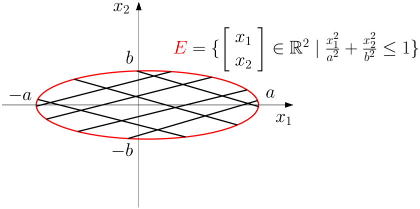
$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.$$





$$E = \{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \mid \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \leq 1 \}$$

Solution.



Solution. We need compute the area of the region

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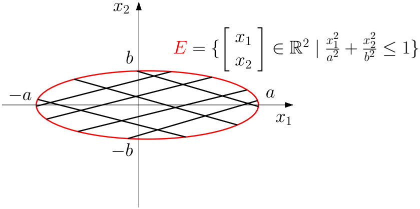

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Consider the unit disk

$$D \quad := \quad \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1 \right\}$$

and the matrix

$$A \quad = \quad \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

Solution (continued). Let $f_A : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation whose standard matrix is A, so that for all $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, we have

$$f_A\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 \\ bx_2 \end{bmatrix}.$$

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We now see that

$$\begin{aligned} f_A[D] &= \left\{ f_A\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1 \right\} \\ &= \left\{ \begin{bmatrix} ax_1 \\ bx_2 \end{bmatrix} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1 \right\} \\ &= \left\{ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2 \mid (\tfrac{y_1}{a})^2 + (\tfrac{y_2}{b})^2 \leq 1 \right\} \\ &= \left\{ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2 \mid \tfrac{y_1^2}{a^2} + \tfrac{y_2^2}{b^2} \leq 1 \right\} \\ &= E. \end{aligned}$$

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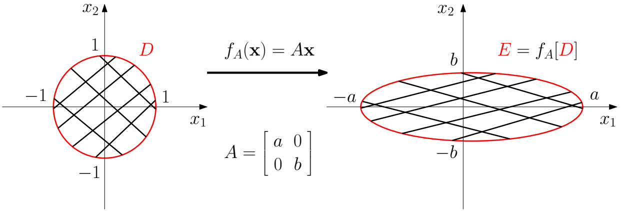
Solution (continued). Reminder: $E = f_A[D]$.



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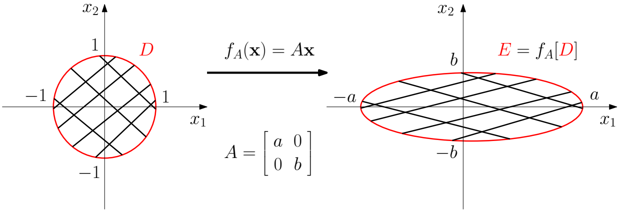

Therefore, the area of E is

$$\text{area}(E) \quad = \quad \underbrace{|\det(A)|}_{=ab} \; \underbrace{\text{area}(D)}_{=1^2\pi} \quad = \quad ab\pi.$$