Linear Algebra 2

Lecture #16
Laplace expansion. Cramer's rule. The adjugate matrix

Irena Penev

March 23, 2023

- Throughout this lecture, $\mathbb{F}$ is a fixed field.

- Throughout this lecture, $\mathbb{F}$ is a fixed field.
- Remark on notation:
  - For a scalar $\alpha \in \mathbb{F} \setminus \{0\}$, we sometimes use the notation $\frac{1}{\alpha}$ instead of α^{-1} (the multiplicative inverse of α in the field $\mathbb{F}$).

- Throughout this lecture, $\mathbb{F}$ is a fixed field.
- Remark on notation:
  - For a scalar $\alpha \in \mathbb{F} \setminus \{0\}$, we sometimes use the notation $\frac{1}{\alpha}$ instead of α^{-1} (the multiplicative inverse of α in the field $\mathbb{F}$).
  - For instance, in $\mathbb{Z}_3$, we have $\frac{1}{1} = 1^{-1} = 1$ and $\frac{1}{2} = 2^{-1} = 2$ (because in $\mathbb{Z}_3$: $2 \cdot 2 = 1$).

- Throughout this lecture, $\mathbb{F}$ is a fixed field.
- Remark on notation:
  - For a scalar $\alpha \in \mathbb{F} \setminus \{0\}$, we sometimes use the notation $\frac{1}{\alpha}$ instead of α^{-1} (the multiplicative inverse of α in the field $\mathbb{F}$).
  - For instance, in $\mathbb{Z}_3$, we have $\frac{1}{1} = 1^{-1} = 1$ and $\frac{1}{2} = 2^{-1} = 2$ (because in $\mathbb{Z}_3$: $2 \cdot 2 = 1$).
  - For scalars $a, b \in \mathbb{F}$ such that $b \neq 0$, we sometimes write $\frac{a}{b}$ instead of $b^{-1}a$.

- Throughout this lecture, $\mathbb{F}$ is a fixed field.

- Remark on notation:
  - For a scalar $\alpha \in \mathbb{F} \setminus \{0\}$, we sometimes use the notation $\frac{1}{\alpha}$ instead of α^{-1} (the multiplicative inverse of α in the field $\mathbb{F}$).
  - For instance, in $\mathbb{Z}_3$, we have $\frac{1}{1} = 1^{-1} = 1$ and $\frac{1}{2} = 2^{-1} = 2$ (because in $\mathbb{Z}_3$: $2 \cdot 2 = 1$).
  - For scalars $a, b \in \mathbb{F}$ such that $b \neq 0$, we sometimes write $\frac{a}{b}$ instead of $b^{-1}a$.
  - For example, in $\mathbb{Z}_5$, we have that $3^{-1} = 2$ (because $3 \cdot 2 = 1$), and so $\frac{4}{3} = 3^{-1} \cdot 4 = 2 \cdot 4 = 3$.

- In this lecture, we continue our study of determinants.

- In this lecture, we continue our study of determinants.
- The lecture has four parts:

- In this lecture, we continue our study of determinants.
- The lecture has four parts:
  1. the multiplicative property of determinants;

- In this lecture, we continue our study of determinants.
- The lecture has four parts:
  1. the multiplicative property of determinants;
  2. Laplace expansion;

- In this lecture, we continue our study of determinants.
- The lecture has four parts:
  1. the multiplicative property of determinants;
  2. Laplace expansion;
  3. Cramer's rule;

- In this lecture, we continue our study of determinants.
- The lecture has four parts:
  1. the multiplicative property of determinants;
  2. Laplace expansion;
  3. Cramer's rule;
  4. the adjugate matrix.

1. The multiplicative property of determinants

1. The multiplicative property of determinants

- Recall that for matrices $A, B \in \mathbb{F}^{n\times n}$ and a scalar $\alpha \in \mathbb{F}$:
  - $\det(A+B) \neq \det(A) + \det(B)$
  - $\det(\alpha A) \neq \alpha \det(A)$.

1. The multiplicative property of determinants

- Recall that for matrices $A, B \in \mathbb{F}^{n\times n}$ and a scalar $\alpha \in \mathbb{F}$:
  - $\det(A + B) \neq \det(A) + \det(B)$
  - $\det(\alpha A) \neq \alpha \det(A)$.

Theorem 1.3

Let $A, B \in \mathbb{F}^{n\times n}$. Then $\det(AB) = \det(A)\det(B)$.

1. The multiplicative property of determinants

- Recall that for matrices $A, B \in \mathbb{F}^{n\times n}$ and a scalar $\alpha \in \mathbb{F}$:
    - $\det(A+B) \neq \det(A) + \det(B)$
    - $\det(\alpha A) \neq \alpha \det(A)$.

Theorem 1.3

Let $A, B \in \mathbb{F}^{n\times n}$. Then $\det(AB) = \det(A)\det(B)$.

- To prove Theorem 1.3, we first need a couple of propositions.

Proposition 1.1

Let $A, B \in \mathbb{F}^{n \times n}$. Then AB is invertible if and only if A and B are both invertible.

Proof.

Proposition 1.1

Let $A, B \in \mathbb{F}^{n\times n}$. Then AB is invertible if and only if A and B are both invertible.

Proof. If A and B are invertible, then it is clear that AB is invertible with inverse $B^{-1}A^{-1}$.

Proposition 1.1

Let $A, B \in \mathbb{F}^{n \times n}$. Then AB is invertible if and only if A and B are both invertible.

Proof. If A and B are invertible, then it is clear that AB is invertible with inverse $B^{-1}A^{-1}$.

Suppose now that AB is invertible. Then $\text{rank}(AB) = n$.

Proposition 1.1

Let $A, B \in \mathbb{F}^{n\times n}$. Then AB is invertible if and only if A and B are both invertible.

Proof. If A and B are invertible, then it is clear that AB is invertible with inverse $B^{-1}A^{-1}$.

Suppose now that AB is invertible. Then $\operatorname{rank}(AB) = n$. But we know that $\operatorname{rank}(AB) \leq \min\left\{\operatorname{rank}(A), \operatorname{rank}(B)\right\}$, and it follows that $\operatorname{rank}(A) = \operatorname{rank}(B) = n$.

Proposition 1.1

Let $A, B \in \mathbb{F}^{n\times n}$. Then AB is invertible if and only if A and B are both invertible.

Proof. If A and B are invertible, then it is clear that AB is invertible with inverse $B^{-1}A^{-1}$.

Suppose now that AB is invertible. Then $\operatorname{rank}(AB) = n$. But we know that $\operatorname{rank}(AB) \leq \min\left\{\operatorname{rank}(A), \operatorname{rank}(B)\right\}$, and it follows that $\operatorname{rank}(A) = \operatorname{rank}(B) = n$. But now A and B are both invertible.

Proposition 1.2

Let $A, E \in \mathbb{F}^{n \times n}$, and assume that E is an elementary matrix. Then $\det(EA) = \det(E)\det(A)$.

Proof.

Proposition 1.2

Let $A, E \in \mathbb{F}^{n \times n}$, and assume that E is an elementary matrix. Then $\det(EA) = \det(E)\det(A)$.

Proof. Let R be an elementary row operation that corresponds to the elementary matrix E, so that both the following hold:

1. E is the matrix obtained by performing R on I_n;
2. EA is the matrix obtained by performing R on A.

Proposition 1.2

Let $A, E \in \mathbb{F}^{n \times n}$, and assume that E is an elementary matrix. Then $\det(EA) = \det(E)\det(A)$.

Proof. Let R be an elementary row operation that corresponds to the elementary matrix E, so that both the following hold:

1. E is the matrix obtained by performing R on I_n;
2. EA is the matrix obtained by performing R on A.

By Theorem 4.2 from Lecture Notes 15, there exists some scalar $\alpha \in \mathbb{F} \setminus \{0\}$ such that for any matrix $M \in \mathbb{F}^{n \times n}$, the determinant of the matrix obtained by performing the elementary row operation R on M is $\alpha\det(M)$.

Proposition 1.2

Let $A, E \in \mathbb{F}^{n\times n}$, and assume that E is an elementary matrix. Then $\det(EA) = \det(E)\det(A)$.

Proof. Let R be an elementary row operation that corresponds to the elementary matrix E, so that both the following hold:

1. E is the matrix obtained by performing R on I_n;
2. EA is the matrix obtained by performing R on A.

By Theorem 4.2 from Lecture Notes 15, there exists some scalar $\alpha \in \mathbb{F} \setminus \{0\}$ such that for any matrix $M \in \mathbb{F}^{n\times n}$, the determinant of the matrix obtained by performing the elementary row operation R on M is $\alpha\det(M)$. So,

1. $\det(E) = \alpha\det(I_n) = \alpha$;
2. $\det(EA) = \alpha\det(A)$.

Proposition 1.2

Let $A, E \in \mathbb{F}^{n\times n}$, and assume that E is an elementary matrix. Then $\det(EA) = \det(E)\det(A)$.

Proof. Let R be an elementary row operation that corresponds to the elementary matrix E, so that both the following hold:

1. E is the matrix obtained by performing R on I_n;
2. EA is the matrix obtained by performing R on A.

By Theorem 4.2 from Lecture Notes 15, there exists some scalar $\alpha \in \mathbb{F} \setminus \{0\}$ such that for any matrix $M \in \mathbb{F}^{n\times n}$, the determinant of the matrix obtained by performing the elementary row operation R on M is $\alpha\det(M)$. So,

1. $\det(E) = \alpha\det(I_n) = \alpha$;
2. $\det(EA) = \alpha\det(A)$.

It follows that $\det(EA) = \alpha\det(A) = \det(E)\det(A)$. Q.E.D.

Proposition 1.1

Let $A, B \in \mathbb{F}^{n \times n}$. Then AB is invertible if and only if A and B are both invertible.

Proposition 1.2

Let $A, E \in \mathbb{F}^{n \times n}$, and assume that E is an elementary matrix. Then $\det(EA) = \det(E)\det(A)$.

Proposition 1.1

Let $A, B \in \mathbb{F}^{n \times n}$. Then AB is invertible if and only if A and B are both invertible.

Proposition 1.2

Let $A, E \in \mathbb{F}^{n \times n}$, and assume that E is an elementary matrix. Then $\det(EA) = \det(E)\det(A)$.

Theorem 1.3

Let $A, B \in \mathbb{F}^{n \times n}$. Then $\det(AB) = \det(A)\det(B)$.

Proof.

Proposition 1.1

Let $A, B \in \mathbb{F}^{n\times n}$. Then AB is invertible if and only if A and B are both invertible.

Proposition 1.2

Let $A, E \in \mathbb{F}^{n\times n}$, and assume that E is an elementary matrix. Then $\det(EA) = \det(E)\det(A)$.

Theorem 1.3

Let $A, B \in \mathbb{F}^{n\times n}$. Then $\det(AB) = \det(A)\det(B)$.

Proof. Suppose first that at least one of A, B is non-invertible.

Proposition 1.1

Let $A, B \in \mathbb{F}^{n\times n}$. Then AB is invertible if and only if A and B are both invertible.

Proposition 1.2

Let $A, E \in \mathbb{F}^{n\times n}$, and assume that E is an elementary matrix. Then $\det(EA) = \det(E)\det(A)$.

Theorem 1.3

Let $A, B \in \mathbb{F}^{n\times n}$. Then $\det(AB) = \det(A)\det(B)$.

Proof. Suppose first that at least one of A, B is non-invertible. Then by Proposition 1.1, AB is also non-invertible.

Proposition 1.1

Let $A, B \in \mathbb{F}^{n\times n}$. Then AB is invertible if and only if A and B are both invertible.

Proposition 1.2

Let $A, E \in \mathbb{F}^{n\times n}$, and assume that E is an elementary matrix. Then $\det(EA) = \det(E)\det(A)$.

Theorem 1.3

Let $A, B \in \mathbb{F}^{n\times n}$. Then $\det(AB) = \det(A)\det(B)$.

Proof. Suppose first that at least one of A, B is non-invertible. Then by Proposition 1.1, AB is also non-invertible. But by Theorem 5.1 from Lecture Notes 15, non-invertible matrices have determinant zero, and so $\det(AB) = 0 = \det(A)\det(B)$.

Theorem 1.3

Let $A, B \in \mathbb{F}^{n\times n}$. Then $\det(AB) = \det(A)\det(B)$.

Proof (continued). From now on, we assume that A and B are both invertible.

Theorem 1.3

Let $A, B \in \mathbb{F}^{n\times n}$. Then $\det(AB) = \det(A)\det(B)$.

Proof (continued). From now on, we assume that A and B are both invertible. Therefore, they can each be written as a product of elementary matrices, say $A = E_1^A \dots E_p^A$ and $B = E_1^B \dots E_q^B$, where $E_1^A, \dots, E_p^A, E_1^B, \dots, E_q^B$ are elementary matrices.

Theorem 1.3

Let $A, B \in \mathbb{F}^{n\times n}$. Then $\det(AB) = \det(A)\det(B)$.

Proof (continued). From now on, we assume that A and B are both invertible. Therefore, they can each be written as a product of elementary matrices, say $A = E_1^A \dots E_p^A$ and $B = E_1^B \dots E_q^B$, where $E_1^A, \dots, E_p^A, E_1^B, \dots, E_q^B$ are elementary matrices. So, $AB = E_1^A \dots E_p^A E_1^B \dots E_q^B$.

Theorem 1.3

Let $A, B \in \mathbb{F}^{n \times n}$. Then $\det(AB) = \det(A)\det(B)$.

Proof (continued). From now on, we assume that A and B are both invertible. Therefore, they can each be written as a product of elementary matrices, say $A = E_1^A \dots E_p^A$ and $B = E_1^B \dots E_q^B$, where $E_1^A, \dots, E_p^A, E_1^B, \dots, E_q^B$ are elementary matrices. So, $AB = E_1^A \dots E_p^A E_1^B \dots E_q^B$. By repeatedly applying Proposition 1.2, we get that

- $\det(A) = \det(E_1^A) \dots \det(E_p^A)$;
- $\det(B) = \det(E_1^B) \dots \det(E_q^B)$;
- $\det(AB) = \det(E_1^A) \dots \det(E_p^A)\det(E_1^B) \dots \det(E_q^B)$.

Theorem 1.3

Let $A, B \in \mathbb{F}^{n\times n}$. Then $\det(AB) = \det(A)\det(B)$.

Proof (continued). From now on, we assume that A and B are both invertible. Therefore, they can each be written as a product of elementary matrices, say $A = E_1^A \dots E_p^A$ and $B = E_1^B \dots E_q^B$, where $E_1^A, \dots, E_p^A, E_1^B, \dots, E_q^B$ are elementary matrices. So, $AB = E_1^A \dots E_p^A E_1^B \dots E_q^B$. By repeatedly applying Proposition 1.2, we get that

- $\det(A) = \det(E_1^A) \dots \det(E_p^A)$;
- $\det(B) = \det(E_1^B) \dots \det(E_q^B)$;
- $\det(AB) = \det(E_1^A) \dots \det(E_p^A)\det(E_1^B) \dots \det(E_q^B)$.

But now

$$\det(AB) = \underbrace{\det(E_1^A) \dots \det(E_p^A)}_{=\det(A)} \underbrace{\det(E_1^B) \dots \det(E_q^B)}_{=\det(B)} = \det(A)\det(B).$$

Q.E.D.

Theorem 1.3

Let $A, B \in \mathbb{F}^{n\times n}$. Then $\det(AB) = \det(A)\det(B)$.

Theorem 1.3

Let $A, B \in \mathbb{F}^{n \times n}$. Then $\det(AB) = \det(A)\det(B)$.

Corollary 1.4

Let $A \in \mathbb{F}^{n \times n}$ be an invertible matrix. Then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Proof.

Theorem 1.3

Let $A, B \in \mathbb{F}^{n\times n}$. Then $\det(AB) = \det(A)\det(B)$.

Corollary 1.4

Let $A \in \mathbb{F}^{n\times n}$ be an invertible matrix. Then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Proof. Since $AA^{-1} = I_n$, we see that

$$\begin{aligned}
\det(A)\det(A^{-1}) &= \det(AA^{-1}) \qquad \text{by Theorem 1.3} \\
&= \det(I_n) \\
&= 1,
\end{aligned}$$

and consequently, $\det(A^{-1}) = \frac{1}{\det(A)}$.

Theorem 1.3

Let $A, B \in \mathbb{F}^{n\times n}$. Then $\det(AB) = \det(A)\det(B)$.

Corollary 1.5

Let A be an orthogonal matrix in $\mathbb{R}^{n\times n}$. Then $\det(A) = \pm 1$.

Proof.

Theorem 1.3

Let $A, B \in \mathbb{F}^{n\times n}$. Then $\det(AB) = \det(A)\det(B)$.

Corollary 1.5

Let A be an orthogonal matrix in $\mathbb{R}^{n\times n}$. Then $\det(A) = \pm 1$.

Proof. Since A is orthogonal, it satisfies $A^T A = I_n$ (by definition).

Theorem 1.3

Let $A, B \in \mathbb{F}^{n \times n}$. Then $\det(AB) = \det(A)\det(B)$.

Corollary 1.5

Let A be an orthogonal matrix in $\mathbb{R}^{n \times n}$. Then $\det(A) = \pm 1$.

Proof. Since A is orthogonal, it satisfies $A^T A = I_n$ (by definition). Therefore,

$$1 = \det(I_n) = \det(A^T A) \overset{(*)}{=} \det(A^T)\det(A) \overset{(**)}{=} \det(A)^2,$$

where (*) follows from Theorem 1.3, and (**) follows from Theorem 2.2 of Lecture Notes 15. But now we see that $\det(A) = \pm 1$. Q.E.D.

Corollary 1.5

Let A be an orthogonal matrix in $\mathbb{R}^{n\times n}$. Then $\det(A) = \pm 1$.

- **Warning:** The converse of Corollary 1.5 is false, i.e. matrices whose determinant is ± 1 need not be orthogonal.

Corollary 1.5

Let A be an orthogonal matrix in $\mathbb{R}^{n \times n}$. Then $\det(A) = \pm 1$.

- **Warning:** The converse of Corollary 1.5 is false, i.e. matrices whose determinant is ± 1 need not be orthogonal.
- For example, the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ satisfies $\det(A) = 1$, but A is not orthogonal.

Corollary 1.5

Let A be an orthogonal matrix in $\mathbb{R}^{n\times n}$. Then $\det(A) = \pm 1$.

- **Warning:** The converse of Corollary 1.5 is false, i.e. matrices whose determinant is ± 1 need not be orthogonal.
- For example, the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ satisfies $\det(A) = 1$, but A is not orthogonal.
- More generally, suppose that A is any invertible matrix in $\mathbb{R}^{n\times n}$, and form the matrix B by multiplying one row or one column of A by $\frac{1}{\det(A)}$. Then $\det(B) = 1$, but B need not be orthogonal.

2. Laplace expansion

- Given a matrix $A = [a_{i,j}]_{n\times n}$ in $\mathbb{F}^{n\times n}$ (where $n \geq 2$) and indices $p, q \in \{1, \dots, n\}$, $A_{p,q}$ is the $(n-1)\times(n-1)$ matrix obtained from A by deleting the p-th row and q-th column.
- For example, for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

we have

$$A_{1,1} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}; \qquad A_{2,1} = \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}; \qquad A_{3,1} = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix};$$

$$A_{1,2} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}; \qquad A_{2,2} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}; \qquad A_{3,2} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix};$$

$$A_{1,3} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}; \qquad A_{2,3} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}; \qquad A_{3,3} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}.$$

- Given a matrix $A = [a_{i,j}]_{n\times n}$ in $\mathbb{F}^{n\times n}$ (where $n \geq 2$) and indices $p, q \in \{1, \ldots, n\}$, $A_{p,q}$ is the $(n-1)\times(n-1)$ matrix obtained from A by deleting the p-th row and q-th column.

- Given a matrix $A = [a_{i,j}]_{n \times n}$ in $\mathbb{F}^{n \times n}$ (where $n \geq 2$) and indices $p, q \in \{1, \dots, n\}$, $A_{p,q}$ is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the p-th row and q-th column.
- For a matrix $A \in \mathbb{F}^{n \times n}$ (with $n \geq 2$):
  - the determinants $\det(A_{i,j})$ (with $i, j \in \{1, \dots, n\}$) are referred to as the *first minors* of A;
  - the numbers $C_{i,j} := (-1)^{i+j} \det(A_{i,j})$ (with $i, j \in \{1, \dots, n\}$) are referred to as the *cofactors* of A.

Laplace expansion

Let $A = [a_{i,j}]_{n \times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n \times n}$. Then both the following hold:

(a) [expansion along the i-th row] for all $i \in \{1, \dots, n\}$:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \, a_{i,j} \, \det(A_{i,j});$$

(b) [expansion along the j-th column] for all $j \in \{1, \dots, n\}$:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \, a_{i,j} \, \det(A_{i,j}).$$

- Laplace expansion is also referred to as "cofactor expansion."
- This is because in terms of cofactors $C_{i,j}$ of A:
  - the formula from (a) becomes $\det(A) = \sum_{j=1}^{n} a_{i,j} C_{i,j}$;
  - the formula from (b) becomes $\det(A) = \sum_{i=1}^{n} a_{i,j} C_{i,j}$.

Laplace expansion

Let $A = [a_{i,j}]_{n\times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n\times n}$. Then both the following hold:

(a) [expansion along the i-th row] for all $i \in \{1, \ldots, n\}$:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j});$$

(b) [expansion along the j-th column] for all $j \in \{1, \ldots, n\}$:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j}).$$

- Proof: Later!

Laplace expansion

Let $A = [a_{i,j}]_{n \times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n \times n}$. Then both the following hold:

(a) [expansion along the i-th row] for all $i \in \{1, \dots, n\}$:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \, a_{i,j} \det(A_{i,j});$$

(b) [expansion along the j-th column] for all $j \in \{1, \dots, n\}$:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \, a_{i,j} \det(A_{i,j}).$$

- Proof: Later!
- First, a couple of examples.

Example 2.2

Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \\ 7 & 0 & 8 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Compute $\det(A)$ in two ways:

(a) via Laplace expansion along the third row;

(b) via Laplace expansion along the second column.

Solution. (a) Laplace expansion along the third row:

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \\ 7 & 0 & 8 \end{vmatrix} \\
&= (-1)^{3+1}7\begin{vmatrix} 0 & 1 \\ 4 & 5 \end{vmatrix} + (-1)^{3+2}0\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} + (-1)^{3+3}8\begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix} \\
&= 7\underbrace{\begin{vmatrix} 0 & 1 \\ 4 & 5 \end{vmatrix}}_{=-4} + 8\underbrace{\begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix}}_{=8} \\
&= 36.
\end{aligned}$$

Solution (continued). (b) Laplace expansion along the second column:

$$
\begin{aligned}
\det(A) &= \begin{vmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \\ 7 & 0 & 8 \end{vmatrix} \\
&= (-1)^{1+2} 0 \begin{vmatrix} 3 & 5 \\ 7 & 8 \end{vmatrix} + (-1)^{2+2} 4 \begin{vmatrix} 2 & 1 \\ 7 & 8 \end{vmatrix} + (-1)^{3+2} 0 \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} \\
&= 4 \underbrace{\begin{vmatrix} 2 & 1 \\ 7 & 8 \end{vmatrix}}_{=9} \\
&= 36.
\end{aligned}
$$

Example 2.2

Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \\ 7 & 0 & 8 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Compute $\det(A)$ in two ways:

(a) via Laplace expansion along the third row;

(b) via Laplace expansion along the second column.

Example 2.2

Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \\ 7 & 0 & 8 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Compute $\det(A)$ in two ways:

(a) via Laplace expansion along the third row;

(b) via Laplace expansion along the second column.

- As a general rule, it is best to choose a row/column with a lot of zeros for expansion (if such a row/column exists).

Example 2.2

Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \\ 7 & 0 & 8 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Compute $\det(A)$ in two ways:

(a) via Laplace expansion along the third row;

(b) via Laplace expansion along the second column.

- As a general rule, it is best to choose a row/column with a lot of zeros for expansion (if such a row/column exists).
- In Example 2.2 (above), (b) was easier than (a).

Example 2.4

Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 1 \\ 3 & -3 & 5 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$.

Solution.

Example 2.4

Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 1 \\ 3 & -3 & 5 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$.

Solution. We combine various methods for computing determinants:

$$\det(A) = \begin{vmatrix} 1 & -1 & 2 \\ -2 & 4 & 1 \\ 3 & -3 & 5 \end{vmatrix} \overset{C_2 \to C_2 + C_1}{=} \begin{vmatrix} 1 & 0 & 2 \\ -2 & 2 & 1 \\ 3 & 0 & 5 \end{vmatrix}$$

$$\overset{(*)}{=} (-1)^{2+2} 2 \underbrace{\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}}_{=-1} = -2.$$

where in (*), we performed Laplace expansion along 2nd column.

Proposition 2.1

Let $A \in \mathbb{F}^{(n-1)\times(n-1)}$ (where $n \geq 2$) and $\mathbf{a} \in \mathbb{F}^{n-1}$. Then

$$\det\left(\begin{bmatrix} A & \mathbf{0} \\ \mathbf{a}^T & 1 \end{bmatrix}_{n\times n} \right) \;=\; \det(A).$$

Proof.

Proposition 2.1

Let $A \in \mathbb{F}^{(n-1)\times(n-1)}$ (where $n \geq 2$) and $\mathbf{a} \in \mathbb{F}^{n-1}$. Then

$$\det\Big(\begin{bmatrix} A & \mathbf{0} \\ \mathbf{a}^T & 1 \end{bmatrix}_{n\times n} \Big) \;=\; \det(A).$$

Proof. First, set $\begin{bmatrix} A & \mathbf{0} \\ \mathbf{a}^T & 1 \end{bmatrix}_{n\times n} = [a_{i,j}]_{n\times n}$, so that:

- $A = [a_{i,j}]_{(n-1)\times(n-1)}$;
- $a_{n,n} = 1$;
- for all $i \in \{1, \dots, n-1\}$, $a_{i,n} = 0$;
- for all $j \in \{1, \dots, n-1\}$, $a_{n,j}$ is the j-th entry of the vector $\mathbf{a}$.

Proposition 2.1

Let $A \in \mathbb{F}^{(n-1)\times(n-1)}$ (where $n \geq 2$) and $\mathbf{a} \in \mathbb{F}^{n-1}$. Then

$$\det\left(\begin{bmatrix} A & \mathbf{0} \\ \mathbf{a}^T & 1 \end{bmatrix}_{n\times n} \right) \;=\; \det(A).$$

Proof (continued). Next, let $S_n^* := \{\sigma \in S_n \mid \sigma(n) = n\}$.

Proposition 2.1

Let $A \in \mathbb{F}^{(n-1)\times(n-1)}$ (where $n \geq 2$) and $\mathbf{a} \in \mathbb{F}^{n-1}$. Then

$$\det\left(\begin{bmatrix} A & \mathbf{0} \\ \mathbf{a}^T & 1 \end{bmatrix}_{n\times n} \right) = \det(A).$$

Proof (continued). Next, let $S_n^* := \{\sigma \in S_n \mid \sigma(n) = n\}$.

For all $\sigma \in S_n^*$, let $\sigma^* : \{1, \ldots, n-1\} \to \{1, \ldots, n-1\}$ be given by $\sigma^*(i) = \sigma(i)$ for all $i \in \{1, \ldots, n-1\}$.

Proposition 2.1

Let $A \in \mathbb{F}^{(n-1)\times(n-1)}$ (where $n \geq 2$) and $\mathbf{a} \in \mathbb{F}^{n-1}$. Then

$$\det\left(\begin{bmatrix} A & \mathbf{0} \\ \mathbf{a}^T & 1 \end{bmatrix}_{n\times n}\right) = \det(A).$$

Proof (continued). Next, let $S_n^* := \{\sigma \in S_n \mid \sigma(n) = n\}$.

For all $\sigma \in S_n^*$, let $\sigma^* : \{1, \dots, n-1\} \to \{1, \dots, n-1\}$ be given by $\sigma^*(i) = \sigma(i)$ for all $i \in \{1, \dots, n-1\}$.

So, for all $\sigma \in S_n^*$, we have that $\sigma^* \in S_{n-1}$, and moreover, we have that $\text{sgn}(\sigma^*) = \text{sgn}(\sigma)$.

Proposition 2.1

Let $A \in \mathbb{F}^{(n-1)\times(n-1)}$ (where $n \geq 2$) and $\mathbf{a} \in \mathbb{F}^{n-1}$. Then

$$\det\Big(\begin{bmatrix} A & \mathbf{0} \\ \mathbf{a}^T & 1 \end{bmatrix}_{n\times n} \Big) = \det(A).$$

Proof (continued). Next, let $S_n^* := \{\sigma \in S_n \mid \sigma(n) = n\}$.

For all $\sigma \in S_n^*$, let $\sigma^* : \{1, \dots, n-1\} \to \{1, \dots, n-1\}$ be given by $\sigma^*(i) = \sigma(i)$ for all $i \in \{1, \dots, n-1\}$.

So, for all $\sigma \in S_n^*$, we have that $\sigma^* \in S_{n-1}$, and moreover, we have that $\text{sgn}(\sigma^*) = \text{sgn}(\sigma)$.

Furthermore, it is clear that the mapping $\sigma \mapsto \sigma^*$ is a bijection from S_n^* to S_{n-1}.

Proposition 2.1

Let $A \in \mathbb{F}^{(n-1)\times(n-1)}$ (where $n \geq 2$) and $\mathbf{a} \in \mathbb{F}^{n-1}$. Then

$$\det\left(\begin{bmatrix} A & \mathbf{0} \\ \mathbf{a}^T & 1 \end{bmatrix}_{n\times n} \right) = \det(A).$$

Proof (continued). Next, let $S_n^* := \{\sigma \in S_n \mid \sigma(n) = n\}$.

For all $\sigma \in S_n^*$, let $\sigma^* : \{1, \dots, n-1\} \to \{1, \dots, n-1\}$ be given by $\sigma^*(i) = \sigma(i)$ for all $i \in \{1, \dots, n-1\}$.

So, for all $\sigma \in S_n^*$, we have that $\sigma^* \in S_{n-1}$, and moreover, we have that $\text{sgn}(\sigma^*) = \text{sgn}(\sigma)$.

Furthermore, it is clear that the mapping $\sigma \mapsto \sigma^*$ is a bijection from S_n^* to S_{n-1}.

We now compute (next slide):

Proof (continued).

$$\begin{aligned}
\det\left(\left[\begin{array}{cc} A & \mathbf{0} \\ \mathbf{a}^T & 1 \end{array}\right]_{n\times n}\right) &= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{n-1,\sigma(n-1)} a_{n,\sigma(n)} \\
&\overset{(*)}{=} \sum_{\sigma\in S_n^*} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{n-1,\sigma(n-1)} \underbrace{a_{n,\sigma(n)}}_{=1} \\
&= \sum_{\sigma\in S_n^*} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{n-1,\sigma(n-1)} \\
&= \sum_{\sigma\in S_n^*} \operatorname{sgn}(\sigma^*) a_{1,\sigma^*(1)} \dots a_{n-1,\sigma^*(n-1)} \\
&= \sum_{\pi\in S_{n-1}} \operatorname{sgn}(\pi) a_{1,\pi(1)} \dots a_{n-1,\pi(n-1)} \\
&= \det(A),
\end{aligned}$$

where (*) follows from the fact that for all $\sigma \in S_n \setminus S_n^*$, we have that $a_{1,\sigma(1)} \dots a_{n-1,\sigma(n-1)} a_{n,\sigma(n)} = 0$. Q.E.D.

Laplace expansion

Let $A = [a_{i,j}]_{n\times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n\times n}$. Then both the following hold:

(a) [expansion along the i-th row] for all $i \in \{1, \dots, n\}$:

$$\det(A) \;=\; \sum_{j=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j});$$

(b) [expansion along the j-th column] for all $j \in \{1, \dots, n\}$:

$$\det(A) \;=\; \sum_{i=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j}).$$

Proof.

Laplace expansion

Let $A = [a_{i,j}]_{n\times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n\times n}$. Then both the following hold:

(a) [expansion along the i-th row] for all $i \in \{1, \dots, n\}$:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j});$$

(b) [expansion along the j-th column] for all $j \in \{1, \dots, n\}$:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j}).$$

Proof. In view of Theorem 2.2 from Lecture Notes 15, it is enough to prove (b).

Laplace expansion

Let $A = [a_{i,j}]_{n\times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n\times n}$. Then both the following hold:

(a) [expansion along the i-th row] for all $i \in \{1,\dots,n\}$:

$$\det(A) \;=\; \sum_{j=1}^{n}(-1)^{i+j}\, a_{i,j}\, \det(A_{i,j});$$

(b) [expansion along the j-th column] for all $j \in \{1,\dots,n\}$:

$$\det(A) \;=\; \sum_{i=1}^{n}(-1)^{i+j}\, a_{i,j}\, \det(A_{i,j}).$$

Proof. In view of Theorem 2.2 from Lecture Notes 15, it is enough to prove (b). Fix $j \in \{1,\dots,n\}$. WTS

$$\det(A) \;=\; \sum_{i=1}^{n}(-1)^{i+j}\, a_{i,j}\, \det(A_{i,j}).$$

Proof (continued). First, set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$.

Proof (continued). First, set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$. Then $\mathbf{a}_j = \sum_{i=1}^{n} a_{i,j}\mathbf{e}_i$, and so

$$\begin{aligned}
\det(A) &= \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_j & \mathbf{a}_{j+1} & \mathbf{a}_n \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \sum_{i=1}^{n} a_{i,j}\mathbf{e}_i & \mathbf{a}_{j+1} & \mathbf{a}_n \end{bmatrix}\right) \\
&\overset{(*)}{=} \sum_{i=1}^{n} a_{i,j}\det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \mathbf{a}_n \end{bmatrix}\right),
\end{aligned}$$

where (*) follows from Proposition 3.1(a) from Lecture Notes 15.

Proof (continued). First, set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$. Then $\mathbf{a}_j = \sum_{i=1}^{n} a_{i,j}\mathbf{e}_i$, and so

$$\begin{aligned} \det(A) &= \det\Big(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_j & \mathbf{a}_{j+1} & \mathbf{a}_n \end{bmatrix} \Big) \\ &= \det\Big(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \sum_{i=1}^{n} a_{i,j}\mathbf{e}_i & \mathbf{a}_{j+1} & \mathbf{a}_n \end{bmatrix} \Big) \\ &\stackrel{(*)}{=} \sum_{i=1}^{n} a_{i,j} \det\Big(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \mathbf{a}_n \end{bmatrix} \Big), \end{aligned}$$

where (*) follows from Proposition 3.1(a) from Lecture Notes 15. Fix an arbitrary index $i \in \{1, \dots, n\}$. To complete the proof, it now suffices to show that

$$\underbrace{\det\Big(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \mathbf{a}_n \end{bmatrix} \Big)}_{:=B_i} = (-1)^{i+j} \det(A_{i,j}).$$

Proof (continued). Reminder: WTS $\det(B_i) = (-1)^{i+j} \det(A_{i,j})$, where $B_i = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \mathbf{a}_n \end{bmatrix}$.

Proof (continued). Reminder: WTS $\det(B_i) = (-1)^{i+j} \det(A_{i,j})$, where $B_i = \begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \mathbf{a}_n \end{bmatrix}$.

By iteratively performing $n - j$ column swaps on B_i, we can obtain the matrix

$$C_i := \begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_{j-1} & \mathbf{a}_{j+1} & \mathbf{a}_n & \mathbf{e}_i \end{bmatrix}.$$

Proof (continued). Reminder: WTS $\det(B_i) = (-1)^{i+j} \det(A_{i,j})$, where $B_i = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \mathbf{a}_n \end{bmatrix}$.

By iteratively performing $n - j$ column swaps on B_i, we can obtain the matrix

$$C_i := \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_{j+1} & \mathbf{a}_n & \mathbf{e}_i \end{bmatrix}.$$

By iteratively performing $n - i$ row swaps on the matrix C_i, we can obtain the matrix

$$\begin{bmatrix} A_{i,j} & \mathbf{0} \\ \mathbf{a}^T & 1 \end{bmatrix},$$

where $\mathbf{a}^T$ is the row vector of length $n - 1$ obtained from the i-th row of A by deleting its j-th entry.

Proof (continued). Reminder: WTS $\det(B_i) = (-1)^{i+j}\det(A_{i,j})$, where $B_i = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \mathbf{a}_n \end{bmatrix}$.

By iteratively performing $n-j$ column swaps on B_i, we can obtain the matrix

$$C_i := \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_{j+1} & \mathbf{a}_n & \mathbf{e}_i \end{bmatrix}.$$

By iteratively performing $n-i$ row swaps on the matrix C_i, we can obtain the matrix

$$\begin{bmatrix} A_{i,j} & \mathbf{0} \\ \mathbf{a}^T & 1 \end{bmatrix},$$

where $\mathbf{a}^T$ is the row vector of length $n-1$ obtained from the i-th row of A by deleting its j-th entry.

Since swapping two rows or two columns has the effect of changing the sign of the determinant, we see that (next slide):

Proof (continued). Reminder: WTS $\det(B_i) = (-1)^{i+j} \det(A_{i,j})$.

$$\begin{aligned}
\det(B_i) &= (-1)^{n-j}\det(C_i) \\
&= (-1)^{n-j}(-1)^{n-i}\det\Big(\begin{bmatrix} A_{i,j} & \mathbf{0} \\ \mathbf{a}^T & 1 \end{bmatrix}\Big) \\
&\overset{(*)}{=} (-1)^{2n-i-j}\det(A_{i,j}) \\
&= (-1)^{i+j}\det(A_{i,j})
\end{aligned}$$

where (*) follows from Proposition 2.1. Q.E.D.

Laplace expansion

Let $A = [a_{i,j}]_{n \times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n \times n}$. Then both the following hold:

(a) [expansion along the i-th row] for all $i \in \{1, \dots, n\}$:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \, a_{i,j} \det(A_{i,j});$$

(b) [expansion along the j-th column] for all $j \in \{1, \dots, n\}$:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \, a_{i,j} \det(A_{i,j}).$$

3. Cramer's rule

3. Cramer's rule

- For a matrix $A \in \mathbb{F}^{n \times n}$, a vector $\mathbf{b} \in \mathbb{F}^n$, and an index $j \in \{1, \ldots, n\}$, we denote by $A_j(\mathbf{b})$ the matrix obtained from A by replacing the j-th column of A with $\mathbf{b}$.

3. Cramer's rule

- For a matrix $A \in \mathbb{F}^{n \times n}$, a vector $\mathbf{b} \in \mathbb{F}^n$, and an index $j \in \{1, \ldots, n\}$, we denote by $A_j(\mathbf{b})$ the matrix obtained from A by replacing the j-th column of A with $\mathbf{b}$.
- For example, if

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix},$$

then

- $A_1(\mathbf{b}) = \begin{bmatrix} 4 & 1 & 1 \\ 5 & 2 & 2 \\ 6 & 0 & 3 \end{bmatrix}$;
- $A_2(\mathbf{b}) = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 5 & 2 \\ 0 & 6 & 3 \end{bmatrix}$;
- $A_3(\mathbf{b}) = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 6 \end{bmatrix}$.

Cramer's rule

Let A be an invertible matrix in $\mathbb{F}^{n \times n}$, and let $\mathbf{b} \in \mathbb{F}^n$. Then the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely

$$\mathbf{x} \quad = \quad \begin{bmatrix} \frac{\det(A_1(\mathbf{b}))}{\det(A)} & \frac{\det(A_2(\mathbf{b}))}{\det(A)} & \dots & \frac{\det(A_n(\mathbf{b}))}{\det(A)} \end{bmatrix}^T .$$

Cramer's rule

Let A be an invertible matrix in $\mathbb{F}^{n\times n}$, and let $\mathbf{b} \in \mathbb{F}^n$. Then the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely

$$\mathbf{x} \quad = \quad \begin{bmatrix} \frac{\det(A_1(\mathbf{b}))}{\det(A)} & \frac{\det(A_2(\mathbf{b}))}{\det(A)} & \dots & \frac{\det(A_n(\mathbf{b}))}{\det(A)} \end{bmatrix}^T.$$

- We first do an example, and then we give the proof.

Cramer's rule

Let A be an invertible matrix in $\mathbb{F}^{n\times n}$, and let $\mathbf{b} \in \mathbb{F}^n$. Then the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely

$$\mathbf{x} = \begin{bmatrix} \frac{\det(A_1(\mathbf{b}))}{\det(A)} & \frac{\det(A_2(\mathbf{b}))}{\det(A)} & \cdots & \frac{\det(A_n(\mathbf{b}))}{\det(A)} \end{bmatrix}^T.$$

- We first do an example, and then we give the proof.

Example 3.1

Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_3$. Solve the matrix-vector equation $A\mathbf{x} = \mathbf{b}$.

Solution. Note that $\det(A) = 2 \neq 0$, and in particular, A is invertible. So, Cramer's rule applies. We compute:

- $\det(A_1(\mathbf{b})) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & 1 \end{vmatrix} = 2;$
- $\det(A_2(\mathbf{b})) = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 1;$
- $\det(A_3(\mathbf{b})) = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0.$

By Cramer's rule, $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} \frac{\det(A_1(\mathbf{b}))}{\det(A)} & \frac{\det(A_2(\mathbf{b}))}{\det(A)} & \frac{\det(A_3(\mathbf{b}))}{\det(A)} \end{bmatrix}^T \\ &= \begin{bmatrix} \frac{2}{2} & \frac{1}{2} & \frac{0}{2} \end{bmatrix}^T \\ &= \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^T. \end{aligned}$$

Cramer's rule

Let A be an invertible matrix in $\mathbb{F}^{n\times n}$, and let $\mathbf{b} \in \mathbb{F}^n$. Then the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely

$$\mathbf{x} = \begin{bmatrix} \frac{\det(A_1(\mathbf{b}))}{\det(A)} & \frac{\det(A_2(\mathbf{b}))}{\det(A)} & \cdots & \frac{\det(A_n(\mathbf{b}))}{\det(A)} \end{bmatrix}^T.$$

Proof.

Cramer's rule

Let A be an invertible matrix in $\mathbb{F}^{n\times n}$, and let $\mathbf{b}\in\mathbb{F}^n$. Then the matrix-vector equation $A\mathbf{x}=\mathbf{b}$ has a unique solution, namely

$$\mathbf{x} \quad = \quad \left[\begin{array}{cccc} \frac{\det(A_1(\mathbf{b}))}{\det(A)} & \frac{\det(A_2(\mathbf{b}))}{\det(A)} & \dots & \frac{\det(A_n(\mathbf{b}))}{\det(A)} \end{array}\right]^T.$$

Proof. Since A is invertible, we know that $A\mathbf{x}=\mathbf{b}$ has a unique solution, namely, $\mathbf{x}=A^{-1}\mathbf{b}$.

Cramer's rule

Let A be an invertible matrix in $\mathbb{F}^{n\times n}$, and let $\mathbf{b} \in \mathbb{F}^n$. Then the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely

$$\mathbf{x} \quad = \quad \begin{bmatrix} \frac{\det(A_1(\mathbf{b}))}{\det(A)} & \frac{\det(A_2(\mathbf{b}))}{\det(A)} & \dots & \frac{\det(A_n(\mathbf{b}))}{\det(A)} \end{bmatrix}^T.$$

Proof. Since A is invertible, we know that $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$. For this solution $\mathbf{x}$, we set $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$. Our goal is to show that

$$\mathbf{x} \quad = \quad \begin{bmatrix} \frac{\det(A_1(\mathbf{b}))}{\det(A)} & \frac{\det(A_2(\mathbf{b}))}{\det(A)} & \dots & \frac{\det(A_n(\mathbf{b}))}{\det(A)} \end{bmatrix}^T.$$

Cramer's rule

Let A be an invertible matrix in $\mathbb{F}^{n\times n}$, and let $\mathbf{b} \in \mathbb{F}^n$. Then the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely

$$\mathbf{x} = \begin{bmatrix} \frac{\det(A_1(\mathbf{b}))}{\det(A)} & \frac{\det(A_2(\mathbf{b}))}{\det(A)} & \dots & \frac{\det(A_n(\mathbf{b}))}{\det(A)} \end{bmatrix}^T.$$

Proof. Since A is invertible, we know that $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$. For this solution $\mathbf{x}$, we set $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$. Our goal is to show that

$$\mathbf{x} = \begin{bmatrix} \frac{\det(A_1(\mathbf{b}))}{\det(A)} & \frac{\det(A_2(\mathbf{b}))}{\det(A)} & \dots & \frac{\det(A_n(\mathbf{b}))}{\det(A)} \end{bmatrix}^T.$$

Fix an index $j \in \{1, \dots, n\}$. We must show that

$$x_j = \frac{\det(A_j(\mathbf{b}))}{\det(A)}.$$

Cramer's rule

Let A be an invertible matrix in $\mathbb{F}^{n\times n}$, and let $\mathbf{b} \in \mathbb{F}^n$. Then the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely

$$\mathbf{x} = \begin{bmatrix} \frac{\det(A_1(\mathbf{b}))}{\det(A)} & \frac{\det(A_2(\mathbf{b}))}{\det(A)} & \dots & \frac{\det(A_n(\mathbf{b}))}{\det(A)} \end{bmatrix}^T.$$

Proof (continued). Reminder: WTS $x_j = \frac{\det(A_j(\mathbf{b}))}{\det(A)}$.

Cramer's rule

Let A be an invertible matrix in $\mathbb{F}^{n\times n}$, and let $\mathbf{b} \in \mathbb{F}^n$. Then the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely

$$\mathbf{x} = \begin{bmatrix} \frac{\det(A_1(\mathbf{b}))}{\det(A)} & \frac{\det(A_2(\mathbf{b}))}{\det(A)} & \dots & \frac{\det(A_n(\mathbf{b}))}{\det(A)} \end{bmatrix}^T.$$

Proof (continued). Reminder: WTS $x_j = \frac{\det(A_j(\mathbf{b}))}{\det(A)}$.

Set $A = [\ \mathbf{a}_1 \ \ \dots \ \ \mathbf{a}_n\ ]$. We compute:

$$\begin{aligned}
\det(A_j(\mathbf{b})) &= \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{b} & \mathbf{a}_{j+1} & \dots \mathbf{a}_n \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & A\mathbf{x} & \mathbf{a}_{j+1} & \dots \mathbf{a}_n \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \sum_{i=1}^n x_i \mathbf{a}_i & \mathbf{a}_{j+1} & \dots \mathbf{a}_n \end{bmatrix}\right) \\
&= \sum_{i=1}^n x_i \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_i & \mathbf{a}_{j+1} & \dots \mathbf{a}_n \end{bmatrix}\right) \\
&= x_j \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_j & \mathbf{a}_{j+1} & \dots \mathbf{a}_n \end{bmatrix}\right) \\
&= x_j \det(A),
\end{aligned}$$

and the result follows by dividing both sides by $\det(A) \neq 0$. Q.E.D.

4. The adjugate matrix

4. The adjugate matrix

- Given a matrix $A \in \mathbb{F}^{n \times n}$ (with $n \geq 2$), with cofactors $C_{i,j} = (-1)^{i+j}\det(A_{i,j})$ (for $i, j \in \{1, \ldots, n\}$), the *cofactor matrix* of A is the matrix $[C_{i,j}]_{n \times n}$.

4. The adjugate matrix

- Given a matrix $A \in \mathbb{F}^{n \times n}$ (with $n \geq 2$), with cofactors $C_{i,j} = (-1)^{i+j} \det(A_{i,j})$ (for $i, j \in \{1, \ldots, n\}$), the *cofactor matrix* of A is the matrix $[C_{i,j}]_{n \times n}$.
- The *adjugate matrix* (also called the *classical adjoint*) of A, denoted by $\mathrm{adj}(A)$, is the transponse of the cofactor matrix of A, i.e.

$$\mathrm{adj}(A) \quad := \quad [C_{i,j}]_{n \times n}{}^{T}.$$

So, the i, j-th entry of $\mathrm{adj}(A)$ is the cofactor $C_{j,i}$ (note the swapping of the indices).

4. The adjugate matrix

- Given a matrix $A \in \mathbb{F}^{n\times n}$ (with $n \geq 2$), with cofactors $C_{i,j} = (-1)^{i+j}\det(A_{i,j})$ (for $i, j \in \{1, \ldots, n\}$), the *cofactor matrix* of A is the matrix $[C_{i,j}]_{n\times n}$.
- The *adjugate matrix* (also called the *classical adjoint*) of A, denoted by $\text{adj}(A)$, is the transpose of the cofactor matrix of A, i.e.

$$\text{adj}(A) \quad := \quad [C_{i,j}]_{n\times n}{}^{T}.$$

  So, the i, j-th entry of $\text{adj}(A)$ is the cofactor $C_{j,i}$ (note the swapping of the indices).

Theorem 4.2

Let A be an invertible matrix in $\mathbb{F}^{n\times n}$. Then

$$A^{-1} \quad = \quad \tfrac{1}{\det(A)}\text{adj}(A).$$

Proof. Later!

Example 4.1

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Compute the cofactor and adjugate matrices of A.

Solution.

Example 4.1

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Compute the cofactor and adjugate matrices of A.

Solution. For all $i, j \in \{1, 2, 3\}$, we let $C_{i,j} = (-1)^{i+j}\det(A_{i,j})$. (So, the $C_{i,j}$'s are the cofactors of A.)

Solution (continued). Reminder: $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

We compute:

- $C_{1,1} = (-1)^{1+1} \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = 6$;
- $C_{1,2} = (-1)^{1+2} \begin{vmatrix} 0 & 2 \\ 0 & 3 \end{vmatrix} = 0$;
- $C_{1,3} = (-1)^{1+3} \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0$;
- $C_{2,1} = (-1)^{2+1} \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = -3$;
- $C_{2,2} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = 3$;
- $C_{2,3} = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$;
- $C_{3,1} = (-1)^{3+1} \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$;
- $C_{3,2} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2$;
- $C_{3,3} = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$.

Solution (continued). So, the cofactor matrix of A is

$$\begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ -3 & 3 & 0 \\ 0 & -2 & 2 \end{bmatrix}.$$

The adjugate matrix of A is the transpose of the cofactor matrix, i.e.

$$\operatorname{adj}(A) = \begin{bmatrix} 6 & -3 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix}.$$

Theorem 4.2

Let A be an invertible matrix in $\mathbb{F}^{n\times n}$. Then

$$A^{-1} \quad = \quad \frac{1}{\det(A)}\operatorname{adj}(A).$$

Example 4.3

Show that the matrix

$$A \quad = \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix},$$

(with entries understood to be in $\mathbb{R}$) is invertible, and using Theorem 4.2, find its inverse A^{-1}.

Solution.

Theorem 4.2

Let A be an invertible matrix in $\mathbb{F}^{n\times n}$. Then

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A).$$

Example 4.3

Show that the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix},$$

(with entries understood to be in $\mathbb{R}$) is invertible, and using Theorem 4.2, find its inverse A^{-1}.

Solution. The matrix A is upper triangular, and so its determinant can be computed by multiplying the entries along the main diagonal. So, $\det(A) = 1 \cdot 2 \cdot 3 = 6$. Since $\det(A) \neq 0$, A is invertible.

Solution (continued). In Example 4.1, we compute the adjugate matrix of A:

$$\text{adj}(A) = \begin{bmatrix} 6 & -3 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix}.$$

So, by Theorem 4.2, we have that

$$\begin{aligned} A^{-1} &= \tfrac{1}{\det(A)}\text{adj}(A) \\ &= \tfrac{1}{6}\begin{bmatrix} 6 & -3 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & -1/3 \\ 0 & 0 & 1/3 \end{bmatrix}. \end{aligned}$$

Theorem 4.2

Let A be an invertible matrix in $\mathbb{F}^{n \times n}$. Then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Proof.

Theorem 4.2

Let A be an invertible matrix in $\mathbb{F}^{n\times n}$. Then

$$A^{-1} \quad = \quad \frac{1}{\det(A)}\operatorname{adj}(A).$$

Proof. Since A is invertible, $\det(A) \neq 0$; and in particular, the expression $\frac{1}{\det(A)}\operatorname{adj}(A)$ is defined.

Theorem 4.2

Let A be an invertible matrix in $\mathbb{F}^{n\times n}$. Then

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A).$$

Proof. Since A is invertible, $\det(A) \neq 0$; and in particular, the expression $\frac{1}{\det(A)}\operatorname{adj}(A)$ is defined.

We will prove the theorem by showing that matrices A^{-1} and $\frac{1}{\det(A)}\operatorname{adj}(A)$ have the same corresponding entries.

Theorem 4.2

Let A be an invertible matrix in $\mathbb{F}^{n\times n}$. Then

$$A^{-1} \quad = \quad \frac{1}{\det(A)}\operatorname{adj}(A).$$

Proof. Since A is invertible, $\det(A) \neq 0$; and in particular, the expression $\frac{1}{\det(A)}\operatorname{adj}(A)$ is defined.

We will prove the theorem by showing that matrices A^{-1} and $\frac{1}{\det(A)}\operatorname{adj}(A)$ have the same corresponding entries.

Fix indices $i, j \in \{1, \ldots, n\}$. By the definition of adj(A), we see that the i, j-th entry of the matrix $\frac{1}{\det(A)}\operatorname{adj}(A)$ is

$$\frac{(-1)^{j+i}\det(A_{j,i})}{\det(A)}.$$

Theorem 4.2

Let A be an invertible matrix in $\mathbb{F}^{n\times n}$. Then

$$A^{-1} \quad = \quad \frac{1}{\det(A)}\text{adj}(A).$$

Proof. Since A is invertible, $\det(A) \neq 0$; and in particular, the expression $\frac{1}{\det(A)}\text{adj}(A)$ is defined.

We will prove the theorem by showing that matrices A^{-1} and $\frac{1}{\det(A)}\text{adj}(A)$ have the same corresponding entries.

Fix indices $i, j \in \{1, \ldots, n\}$. By the definition of adj(A), we see that the i, j-th entry of the matrix $\frac{1}{\det(A)}\text{adj}(A)$ is

$$\frac{(-1)^{j+i}\det(A_{j,i})}{\det(A)}.$$

We will use Cramer's rule to show that this is also the i, j-th entry of the matrix A^{-1}.

Proof (continued). Reminder: WTS $\frac{(-1)^{j+i}\det(A_{j,i})}{\det(A)}$ is the i, j-th entry of A^{-1}.

Proof (continued). Reminder: WTS $\frac{(-1)^{j+i}\det(A_{j,i})}{\det(A)}$ is the i,j-th entry of A^{-1}.

Set $A^{-1} = \left[\begin{array}{cc} \mathbf{a}_1^* & \ldots \mathbf{a}_n^* \end{array}\right]$. Since $AA^{-1} = I_n$, we have that

$$A\left[\begin{array}{cc} \mathbf{a}_1^* & \ldots \mathbf{a}_n^* \end{array}\right] = \left[\begin{array}{ccc} \mathbf{e}_1 & \ldots & \mathbf{e}_n \end{array}\right],$$

and consequently,

$$\left[\begin{array}{ccc} A\mathbf{a}_1^* & \ldots & A\mathbf{a}_n^* \end{array}\right] = \left[\begin{array}{ccc} \mathbf{e}_1 & \ldots & \mathbf{e}_n \end{array}\right].$$

Proof (continued). Reminder: WTS $\frac{(-1)^{j+i}\det(A_{j,i})}{\det(A)}$ is the i, j-th entry of A^{-1}.

Set $A^{-1} = \begin{bmatrix} \mathbf{a}_1^* & \dots \mathbf{a}_n^* \end{bmatrix}$. Since $AA^{-1} = I_n$, we have that

$$A\begin{bmatrix} \mathbf{a}_1^* & \dots \mathbf{a}_n^* \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \dots & \mathbf{e}_n \end{bmatrix},$$

and consequently,

$$\begin{bmatrix} A\mathbf{a}_1^* & \dots & A\mathbf{a}_n^* \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \dots & \mathbf{e}_n \end{bmatrix}.$$

In particular, the two matrices above have the same j-th column, and so $A\mathbf{a}_j^* = \mathbf{e}_j$, i.e. $\mathbf{a}_j^*$ is the solution of the equation $A\mathbf{x} = \mathbf{e}_j$ (this solution is unique because A is invertible). So, by Cramer's rule, we have that:

$$\mathbf{a}_j^* = \begin{bmatrix} \frac{\det(A_1(\mathbf{e}_j))}{\det(A)} & \dots & \frac{\det(A_n(\mathbf{e}_j))}{\det(A)} \end{bmatrix}^T.$$

Proof (continued). Reminder: $A^{-1} = \begin{bmatrix} \mathbf{a}_1^* & \dots \mathbf{a}_n^* \end{bmatrix}$;
$\mathbf{a}_j^* = \begin{bmatrix} \frac{\det(A_1(\mathbf{e}_j))}{\det(A)} & \dots & \frac{\det(A_n(\mathbf{e}_j))}{\det(A)} \end{bmatrix}^T$; WTS $\frac{(-1)^{j+i}\det(A_{j,i})}{\det(A)}$ is the i, j-th entry of A^{-1}.

Proof (continued). Reminder: $A^{-1} = \begin{bmatrix} \mathbf{a}_1^* & \dots \mathbf{a}_n^* \end{bmatrix}$;
$\mathbf{a}_j^* = \begin{bmatrix} \frac{\det(A_1(\mathbf{e}_j))}{\det(A)} & \dots & \frac{\det(A_n(\mathbf{e}_j))}{\det(A)} \end{bmatrix}^T$; WTS $\frac{(-1)^{j+i}\det(A_{j,i})}{\det(A)}$ is the i, j-th entry of A^{-1}.

The i-th entry of $\mathbf{a}_j^*$ is

$$\frac{\det(A_i(\mathbf{e}_j))}{\det(A)}.$$

Proof (continued). Reminder: $A^{-1} = \left[\begin{array}{cc} \mathbf{a}_1^* & \dots \mathbf{a}_n^* \end{array} \right]$; $\mathbf{a}_j^* = \left[\begin{array}{ccc} \frac{\det(A_1(\mathbf{e}_j))}{\det(A)} & \dots & \frac{\det(A_n(\mathbf{e}_j))}{\det(A)} \end{array} \right]^T$; WTS $\frac{(-1)^{j+i}\det(A_{j,i})}{\det(A)}$ is the i, j-th entry of A^{-1}.

The i-th entry of $\mathbf{a}_j^*$ is

$$\frac{\det(A_i(\mathbf{e}_j))}{\det(A)}.$$

By Laplace explansion along the i-th column, we get that

$$\det(A_i(\mathbf{e}_j)) \quad = \quad (-1)^{j+i}\det(A_{j,i}).$$

Proof (continued). Reminder: $A^{-1} = \left[\begin{array}{cc} \mathbf{a}_1^* & \dots \mathbf{a}_n^* \end{array}\right]$;
$\mathbf{a}_j^* = \left[\begin{array}{ccc} \frac{\det(A_1(\mathbf{e}_j))}{\det(A)} & \dots & \frac{\det(A_n(\mathbf{e}_j))}{\det(A)} \end{array}\right]^T$; WTS $\frac{(-1)^{j+i}\det(A_{j,i})}{\det(A)}$ is the i, j-th entry of A^{-1}.

The i-th entry of $\mathbf{a}_j^*$ is

$$\frac{\det(A_i(\mathbf{e}_j))}{\det(A)}.$$

By Laplace explansion along the i-th column, we get that

$$\det(A_i(\mathbf{e}_j)) \;\; = \;\; (-1)^{j+i}\det(A_{j,i}).$$

So, the i-th entry of $\mathbf{a}_j^*$ (which is precisely the i, j-th entry of A^{-1}) is

$$\frac{(-1)^{j+i}\det(A_{j,i})}{\det(A)}.$$

Q.E.D.

Theorem 4.2

Let A be an invertible matrix in $\mathbb{F}^{n \times n}$. Then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Theorem 4.2

Let A be an invertible matrix in $\mathbb{F}^{n \times n}$. Then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Corollary 4.4

Let $a, b, c, d \in \mathbb{F}$. Then the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad \neq bc$, and in this case, the inverse of A is given by the formula

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Proof. By Theorem 5.1 from Lecture Notes 15, we know that A is invertible if and only if $\det(A) \neq 0$.

Proof. By Theorem 5.1 from Lecture Notes 15, we know that A is invertible if and only if $\det(A) \neq 0$. Clearly, $\det(A) = ad - bc$, and it follows that A is invertible if and only if $ad - bc \neq 0$, i.e. if and only if $ad \neq bc$.

Proof. By Theorem 5.1 from Lecture Notes 15, we know that A is invertible if and only if $\det(A) \neq 0$. Clearly, $\det(A) = ad - bc$, and it follows that A is invertible if and only if $ad - bc \neq 0$, i.e. if and only if $ad \neq bc$.

Now, assume that A is invertible, so that $ad \neq bc$. We first compute the cofactors $C_{i,j}$ of A:

- $C_{1,1} = (-1)^{1+1}\det(A_{1,1}) = d$;
- $C_{1,2} = (-1)^{1+2}\det(A_{1,2}) = -c$;
- $C_{2,1} = (-1)^{2+1}\det(A_{2,1}) = -b$;
- $C_{2,2} = (-1)^{2+2}\det(A_{2,2}) = a$.

Proof. By Theorem 5.1 from Lecture Notes 15, we know that A is invertible if and only if $\det(A) \neq 0$. Clearly, $\det(A) = ad - bc$, and it follows that A is invertible if and only if $ad - bc \neq 0$, i.e. if and only if $ad \neq bc$.

Now, assume that A is invertible, so that $ad \neq bc$. We first compute the cofactors $C_{i,j}$ of A:

- $C_{1,1} = (-1)^{1+1}\det(A_{1,1}) = d$;
- $C_{1,2} = (-1)^{1+2}\det(A_{1,2}) = -c$;
- $C_{2,1} = (-1)^{2+1}\det(A_{2,1}) = -b$;
- $C_{2,2} = (-1)^{2+2}\det(A_{2,2}) = a$.

The cofactor matrix of A is

$$\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.$$

Proof (continued). The adjugate matrix of A is the transpose of the cofactor matrix, i.e.

$$\mathrm{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Proof (continued). The adjugate matrix of A is the transpose of the cofactor matrix, i.e.

$$\mathrm{adj}(A) \;=\; \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

By Theorem 4.2, we now have that

$$A^{-1} \;=\; \tfrac{1}{\det(A)}\mathrm{adj}(A) \;=\; \tfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

Q.E.D.

Theorem 4.2

Let A be an invertible matrix in $\mathbb{F}^{n\times n}$. Then

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A).$$

Corollary 4.4

Let $a, b, c, d \in \mathbb{F}$. Then the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad \neq bc$, and in this case, the inverse of A is given by the formula

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$