Linear Algebra 2: Lecture 16

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Throughout this lecture, $\mathbb{F}$ is a fixed field.

Remark on notation: For a scalar $\alpha \in \mathbb{F} \setminus \{0\}$, we sometimes use the notation $\frac{1}{\alpha}$ instead of α^{-1} (the multiplicative inverse of α in the field $\mathbb{F}$). For instance, in $\mathbb{Z}_3$, we have $\frac{1}{1} = 1^{-1} = 1$ and $\frac{1}{2} = 2^{-1} = 2$ (because in $\mathbb{Z}_3$, we have that $2 \cdot 2 = 1$). In a similar vein, for scalars $a, b \in \mathbb{F}$ such that $b \neq 0$, we sometimes write $\frac{a}{b}$ instead of $b^{-1}a$. For example, in $\mathbb{Z}_5$, we have that $3^{-1} = 2$ (because $3 \cdot 2 = 1$), and so $\frac{4}{3} = 3^{-1} \cdot 4 = 2 \cdot 4 = 3$.

1 The multiplicative property of determinants

In general, for matrices $A, B \in \mathbb{F}^{n \times n}$ and a scalar $\alpha \in \mathbb{F}$, we have that

$$\det(A+B) \;\; \xcancel{=} \;\; \det(A) + \det(B) \qquad \text{and} \qquad \det(\alpha A) \;\; \xcancel{=} \;\; \alpha \det(A).$$

However, as we shall see (see Theorem 1.3 below), we do have that

$$\det(AB) \;\; = \;\; \det(A)\det(B).$$

We begin with a couple of useful propositions.

Proposition 1.1. *Let $A, B \in \mathbb{F}^{n \times n}$. Then AB is invertible if and only if A and B are both invertible.*

Proof. If A and B are invertible, then it is clear that AB is invertible with inverse $B^{-1}A^{-1}$. Suppose now that AB is invertible. Then $\text{rank}(AB) = n$.[1] But we know that $\text{rank}(AB) \leq \min\Big\{\text{rank}(A), \text{rank}(B)\Big\}$,[2] and it follows that $\text{rank}(A) = \text{rank}(B) = n$. But now A and B are both invertible.[3] $\square$

Proposition 1.2. *Let $A, E \in \mathbb{F}^{n \times n}$, and assume that E is an elementary matrix. Then $\det(EA) = \det(E)\det(A)$.*

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[1] This follows from Theorem 4.1 from Lecture Notes 7.

[2] This was Problem 1(b) from HW#9 from Linear Algebra 1 (winter 2022).

[3] Again, this follows from Theorem 4.1 from Lecture Notes 7.

Proof. Let R be an elementary row operation that corresponds to the elementary matrix E, so that both the following hold:

1. E is the matrix obtained by performing R on I_n;[4]
2. EA is the matrix obtained by performing R on A.

By Theorem 4.2 from Lecture Notes 15, there exists some scalar $\alpha \in \mathbb{F} \setminus \{0\}$ such that for any matrix $M \in \mathbb{F}^{n\times n}$, the determinant of the matrix obtained by performing the elementary row operation R on M is $\alpha\text{det}(M)$. So,

1. $\text{det}(E) = \alpha\text{det}(I_n) = \alpha$;
2. $\text{det}(EA) = \alpha\text{det}(A)$.

It follows that

$$\text{det}(EA) \quad = \quad \alpha\text{det}(A) \quad = \quad \text{det}(E)\text{det}(A),$$

which is what we needed to show. $\square$

We are now ready to prove the multiplicative property of determinants.

Theorem 1.3. *Let $A, B \in \mathbb{F}^{n\times n}$. Then det$(AB)$ = det(A)det(B).*

Proof. Suppose first that at least one of A, B is non-invertible. Then by Proposition 1.1, AB is also non-invertible. But by Theorem 5.1 from Lecture Notes 15, non-invertible matrices have determinant zero, and so $\text{det}(AB) = 0 = \text{det}(A)\text{det}(B)$.[5]

From now on, we assume that A and B are both invertible. Therefore, they can each be written as a product of elementary matrices,[6] say $A = E_1^A \dots E_p^A$ and $B = E_1^B \dots E_q^B$, where $E_1^A, \dots, E_p^A, E_1^B, \dots, E_q^B$ are elementary matrices. So, $AB = E_1^A \dots E_p^A E_1^B \dots E_q^B$. By repeatedly applying Proposition 1.2, we get that

- $\text{det}(A) = \text{det}(E_1^A) \dots \text{det}(E_p^A)$;[7]

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[4] Here, it is possible that $E = I_n$. In this case, we can take R to be the multiplication of the first row by the scalar 1.

[5] If A is non-invertible, then $\text{det}(A) = 0$, and if B is non-invertible, then $\text{det}(B) = 0$. In either case $\text{det}(A)\text{det}(B) = 0$.

[6] This follows from Corollary 5.1 of Lecture Notes 4.

[7] Indeed, by repeatedly applying Proposition 1.2, we see that

$$\begin{aligned} \text{det}(A) &= \text{det}(E_1^A \dots E_p^A) \\ &= \text{det}(E_1^A)\text{det}(E_2^A \dots E_p^A) \\ &= \text{det}(E_1^A)\text{det}(E_2^A)\text{det}(E_3^A \dots E_p^A) \\ &\vdots \\ &= \text{det}(E_1^A) \dots \text{det}(E_p^A). \end{aligned}$$

- $\det(B) = \det(E_1^B) \dots \det(E_q^B)$;
- $\det(AB) = \det(E_1^A) \dots \det(E_p^A)\det(E_1^B) \dots \det(E_q^B)$.

But now

$$\begin{aligned} \det(AB) &= \underbrace{\det(E_1^A) \dots \det(E_p^A)}_{=\det(A)} \underbrace{\det(E_1^B) \dots \det(E_q^B)}_{=\det(B)} \\ &= \det(A)\det(B), \end{aligned}$$

which is what we needed to show. ☐

Corollary 1.4. *Let $A \in \mathbb{F}^{n \times n}$ be an invertible matrix. Then*

$$det(A^{-1}) = \tfrac{1}{det(A)}.$$

Proof. Since $AA^{-1} = I_n$, we see that

$$\begin{aligned} \det(A)\det(A^{-1}) &= \det(AA^{-1}) \qquad \text{by Theorem 1.3} \\ &= \det(I_n) \\ &= 1, \end{aligned}$$

and consequently, $\det(A^{-1}) = \frac{1}{\det(A)}$. ☐

Corollary 1.5. *Let A be an orthogonal matrix in $\mathbb{R}^{n \times n}$. Then $det(A) = \pm 1$ (i.e. $det(A)$ is either $+1$ or -1).*

Proof. Since A is orthogonal, it satisfies $A^T A = I_n$ (by definition). Therefore,

$$1 = \det(I_n) = \det(A^T A) \overset{(*)}{=} \det(A^T)\det(A) \overset{(**)}{=} \det(A)^2,$$

where (*) follows from Theorem 1.3, and (**) follows from Theorem 2.2 of Lecture Notes 15. But now we see that $\det(A) = \pm 1$, which is what we needed to show. ☐

Warning: The converse of Corollary 1.5 is false, i.e. matrices whose determinant is ± 1 need not be orthogonal. For example, the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ satisfies $\det(A) = 1$, but A is not orthogonal. More generally, suppose that A is any invertible matrix in $\mathbb{R}^{n \times n}$. Then by Theorem 5.1 of Lecture Notes 15, we have that $\det(A) \neq 0$. We now form the matrix B by multiplying one row or one column of A by $\frac{1}{\det(A)}$, and we see that $\det(B) = 1$. However, B need not be orthogonal.

2 Laplace expansion

Our goal in this section is to prove a formula ("Laplace explansion") for the determinant of a square matrix in terms of determinants of smaller matrices. We begin with a technical proposition.

Proposition 2.1. *Let* $A \in \mathbb{F}^{(n-1)\times(n-1)}$ *(where* $n \geq 2$*) and* $\mathbf{a} \in \mathbb{F}^{n-1}$*. Then*

$$det\Big(\begin{bmatrix} A & \mathbf{0} \\ \mathbf{a}^T & 1 \end{bmatrix}_{n\times n} \Big) \quad = \quad det(A).$$

Proof. First, set $\begin{bmatrix} A & \mathbf{0} \\ \mathbf{a}^T & 1 \end{bmatrix}_{n\times n} = [a_{i,j}]_{n\times n}$, so that all the following hold:

- $A = [a_{i,j}]_{(n-1)\times(n-1)}$;
- $a_{n,n} = 1$;
- for all $i \in \{1, \dots, n-1\}$, $a_{i,n} = 0$;
- for all $j \in \{1, \dots, n-1\}$, $a_{n,j}$ is the j-th entry of the vector $\mathbf{a}$.

Next, let S_n^* be the set of all permutations $\sigma \in S_n$ such that $\sigma(n) = n$, and for all $\sigma \in S_n^*$, let $\sigma^* : \{1, \dots, n-1\} \to \{1, \dots, n-1\}$ be given by $\sigma^*(i) = \sigma(i)$ for all $i \in \{1, \dots, n-1\}$. So, for all $\sigma \in S_n^*$, we have that $\sigma^* \in S_{n-1}$, and moreover, we have that $\text{sgn}(\sigma^*) = \text{sgn}(\sigma)$.[8] Furthermore, it is clear that the mapping $\sigma \mapsto \sigma^*$ is a bijection from S_n^* to S_{n-1}. We now compute:

$$\begin{aligned}
\det\Big(\begin{bmatrix} A & \mathbf{0} \\ \mathbf{a}^T & 1 \end{bmatrix}_{n\times n} \Big) &= \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{n-1,\sigma(n-1)} a_{n,\sigma(n)} \\
&\overset{(*)}{=} \sum_{\sigma \in S_n^*} \text{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{n-1,\sigma(n-1)} \underbrace{a_{n,\sigma(n)}}_{=1} \\
&= \sum_{\sigma \in S_n^*} \text{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{n-1,\sigma(n-1)} \\
&= \sum_{\sigma \in S_n^*} \text{sgn}(\sigma^*) a_{1,\sigma^*(1)} \dots a_{n-1,\sigma^*(n-1)} \\
&= \sum_{\pi \in S_{n-1}} \text{sgn}(\pi) a_{1,\pi(1)} \dots a_{n-1,\pi(n-1)} \\
&= \det(A),
\end{aligned}$$

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[8] This follows from the definition of the sign of a permutation. Indeed, fix any $\sigma \in S_n^*$, and suppose that the disjoint cycle decomposition of σ has k cycles (when all cycles of length one are included); clearly, one of those cycles is (n). Moreover, the disjoint cycle decomposition of σ^* is obtained from the disjoint cycle decomposition of σ by deleting the cycle (n); so, the disjoint cycle decomposition of σ^* has $k-1$ cycles (when all cycles of length one are included). But now $\text{sgn}(\sigma) = (-1)^{n-k} = (-1)^{(n-1)-(k-1)} = \text{sgn}(\sigma^*)$.

where (*) follows from the fact that for all $\sigma \in S_n \setminus S_n^*$, we have that $a_{1,\sigma(1)} \cdots a_{n-1,\sigma(n-1)} a_{n,\sigma(n)} = 0$.[9] $\square$

We now introduce some terminology and notation. Given a matrix $A = [a_{i,j}]_{n \times n}$ in $\mathbb{F}^{n \times n}$ (where $n \geq 2$) and indices $p, q \in \{1, \dots, n\}$, $A_{p,q}$ is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the p-th row and q-th column. For example, for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

we have

- $A_{1,1} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$;
- $A_{2,1} = \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}$;
- $A_{3,1} = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$;
- $A_{1,2} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$;
- $A_{2,2} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$;
- $A_{3,2} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$;
- $A_{1,3} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$;
- $A_{2,3} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$;
- $A_{3,3} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$.

For a matrix $A \in \mathbb{F}^{n \times n}$ (with $n \geq 2$), the determinants $\det(A_{i,j})$ (with $i, j \in \{1, \dots, n\}$) are referred to as the *first minors* of A, whereas numbers $C_{i,j} := (-1)^{i+j} \det(A_{i,j})$ (with $i, j \in \{1, \dots, n\}$) are referred to as the *cofactors* of A.

We now prove a recursive formula for computing determinants in terms of minors or cofactors. It allows us to compute the determinant of a square matrix in terms of determinants of smaller square matrices. This formula is called "Laplace expansion" or "cofactor expansion."

Laplace expansion. *Let $A = [a_{i,j}]_{n \times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n \times n}$. Then both the following hold:*

(a) [expansion along the i-th row] for all $i \in \{1, \dots, n\}$, we have that

$$det(A) = \sum_{j=1}^{n} (-1)^{i+j} \, a_{i,j} \, det(A_{i,j});$$

(b) [expansion along the j-th column] for all $j \in \{1, \dots, n\}$, we have that

$$det(A) = \sum_{i=1}^{n} (-1)^{i+j} \, a_{i,j} \, det(A_{i,j}).$$

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[9] This is because for any $\sigma \in S_n \setminus S_n^*$, there exists some $i \in \{1, \dots, n-1\}$ such that $\sigma(i) = n$, and then we have that $a_{i,\sigma(i)} = a_{i,n} = 0$.

Remark: If we write $C_{i,j} := (-1)^{i+j}\det(A_{i,j})$ for all $i, j \in \{1, \dots, n\}$ (so, the $C_{i,j}$'s are the cofactors of A), then the formula from (a) becomes $\det(A) = \sum\limits_{j=1}^{n} a_{i,j}C_{i,j}$, and the formula from (b) becomes $\det(A) = \sum\limits_{i=1}^{n} a_{i,j}C_{i,j}$. This is why Laplace expansion is also referred to as "cofactor expansion."

Proof. In view of Theorem 2.2 from Lecture Notes 15, it is enough to prove (b). Fix $j \in \{1, \dots, n\}$. We must show that

$$\det(A) \quad = \quad \sum_{i=1}^{n} (-1)^{i+j}\; a_{i,j}\; \det(A_{i,j}).$$

First, set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$. Then $\mathbf{a}_j = \sum\limits_{i=1}^{n} a_{i,j}\mathbf{e}_i$, and so

$$\begin{aligned}
\det(A) \quad &= \quad \det\Big(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_j & \mathbf{a}_{j+1} & \mathbf{a}_n \end{bmatrix}\Big) \\
&= \quad \det\Bigg(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \sum\limits_{i=1}^{n} a_{i,j}\mathbf{e}_i & \mathbf{a}_{j+1} & \mathbf{a}_n \end{bmatrix}\Bigg) \\
&\overset{(*)}{=} \quad \sum_{i=1}^{n} a_{i,j}\det\Big(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \mathbf{a}_n \end{bmatrix}\Big),
\end{aligned}$$

where (*) follows from Proposition 3.1(a) from Lecture Notes 15. Fix an arbitrary index $i \in \{1, \dots, n\}$. To complete the proof, it now suffices to show that

$$\det\Big(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \mathbf{a}_n \end{bmatrix}\Big) \quad = \quad (-1)^{i+j}\;\det(A_{i,j}).$$

By iteratively performing $n - j$ column swaps on the matrix

$$B_i \quad := \quad \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \mathbf{a}_n \end{bmatrix},$$

we can obtain the matrix

$$C_i := \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_{j+1} & \mathbf{a}_n & \mathbf{e}_i \end{bmatrix}.$$

By iteratively performing $n - i$ row swaps on the matrix C_i, we can obtain the matrix

$$\begin{bmatrix} A_{i,j} & \mathbf{0} \\ \mathbf{a}^T & 1 \end{bmatrix},$$

where $\mathbf{a}^T$ is the row vector of length $n - 1$ obtained from the i-th row of A by deleting its j-th entry.[10] Since swapping two rows or two columns has

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[10] So, $\mathbf{a}^T = \begin{bmatrix} a_{i,1} & \dots & a_{i,j-1} & a_{i,j+1} & \dots & a_{i,n} \end{bmatrix}^T$, and in particular $\mathbb{F}^{n-1}$.

the effect of changing the sign of the determinant, we see that

$$\begin{aligned}\det(B_i) &= (-1)^{n-j}\det(C_i) \\ &= (-1)^{n-j}(-1)^{n-i}\det\Big(\begin{bmatrix} A_{i,j} & \mathbf{0} \\ \mathbf{a}^T & 1 \end{bmatrix}\Big) \\ &\overset{(*)}{=} (-1)^{2n-i-j}\det(A_{i,j}) \\ &= (-1)^{i+j}\det(A_{i,j})\end{aligned}$$

where (*) follows from Proposition 2.1. This completes the argument. ☐

Example 2.2. *Consider the matrix*

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \\ 7 & 0 & 8 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$*. Compute det*(A) *in two ways:*

(a) via Laplace expansion along the third row;

(b) via Laplace expansion along the second column.

Solution. (a) We compute:

$$\begin{aligned}\det(A) &= \begin{vmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \\ 7 & 0 & 8 \end{vmatrix} \\ &= (-1)^{3+1}7\begin{vmatrix} 0 & 1 \\ 4 & 5 \end{vmatrix} + (-1)^{3+2}0\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} + (-1)^{3+3}8\begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix} \\ &= 7\underbrace{\begin{vmatrix} 0 & 1 \\ 4 & 5 \end{vmatrix}}_{=-4} + 8\underbrace{\begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix}}_{=8} \\ &= 36.\end{aligned}$$

(b) We compute:

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \\ 7 & 0 & 8 \end{vmatrix} \\
&= (-1)^{1+2}0 \begin{vmatrix} 3 & 5 \\ 7 & 8 \end{vmatrix} + (-1)^{2+2}4 \begin{vmatrix} 2 & 1 \\ 7 & 8 \end{vmatrix} + (-1)^{3+2}0 \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} \\
&= 4 \underbrace{\begin{vmatrix} 2 & 1 \\ 7 & 8 \end{vmatrix}}_{=9} \\
&= 36.
\end{aligned}$$

☐

Exercise 2.3. *Compute the determinant of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 & -2 \\ 3 & -4 & 0 & -2 & -1 \\ 1 & 2 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 2 \\ 2 & -1 & 0 & 1 & 3 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$.

Solution. As a general rule, it is best to expand along a row or column that has a lot of zeros (if such a row or column exists), since that reduces the amount of calculation that we need to perform. In the calculation below, the row or column along which we are about to expand is in red (to facilitate reading).

$$
\begin{aligned}
\det(A) &= \begin{vmatrix} 1 & 2 & 0 & -1 & -2 \\ 3 & -4 & 0 & -2 & -1 \\ 1 & 2 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 2 \\ 2 & -1 & 0 & 1 & 3 \end{vmatrix} \\
&= (-1)^{3+3}2 \begin{vmatrix} 1 & 2 & -1 & -2 \\ 3 & -4 & -2 & -1 \\ 1 & 0 & 0 & 2 \\ 2 & -1 & 1 & 3 \end{vmatrix} && \text{expansion along 3rd column} \\
&= 2 \begin{vmatrix} 1 & 2 & -1 & -2 \\ 3 & -4 & -2 & -1 \\ 1 & 0 & 0 & 2 \\ 2 & -1 & 1 & 3 \end{vmatrix} \\
&= 2\Bigg((-1)^{3+1}1 \begin{vmatrix} 2 & -1 & -2 \\ -4 & -2 & -1 \\ -1 & 1 & 3 \end{vmatrix} + \Bigg) && \text{expansion along 3rd row} \\
&\quad +(-1)^{3+4}2 \begin{vmatrix} 1 & 2 & -1 \\ 3 & -4 & -2 \\ 2 & -1 & 1 \end{vmatrix} \Bigg) \\
&= 2\Bigg(\underbrace{\begin{vmatrix} 2 & -1 & -2 \\ -4 & -2 & -1 \\ -1 & 1 & 3 \end{vmatrix}}_{=-11} -2 \underbrace{\begin{vmatrix} 1 & 2 & -1 \\ 3 & -4 & -2 \\ 2 & -1 & 1 \end{vmatrix}}_{-25}\Bigg) \\
&= 78,
\end{aligned}
$$

where the determinants of the two 3×3 matrices from the second-to-last line can be obtained in various ways: Laplace expansion, elementary row/column operations, or our diagram for computing determinants of 3×3 matrices (described in section 2 of Lecture Notes 15). ☐

Example 2.4. *Compute the determinant of the matrix*

$$
A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 1 \\ 3 & -3 & 5 \end{bmatrix},
$$

with entries understood to be in $\mathbb{R}$*.*

Solution. We combine various methods for computing determinants, as follows:

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 1 & -1 & 2 \\ -2 & 4 & 1 \\ 3 & -3 & 5 \end{vmatrix} \\
&\overset{C_2 \to C_2 + C_1}{=} \begin{vmatrix} 1 & 0 & 2 \\ -2 & 2 & 1 \\ 3 & 0 & 5 \end{vmatrix} \\
&= (-1)^{2+2} 2 \underbrace{\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}}_{=-1} \qquad \text{expansion along 2nd column} \\
&= -2.
\end{aligned}$$

□

3 Cramer's rule

Before stating Cramer's rule, we set up some notation. For a matrix $A \in \mathbb{F}^{n \times n}$, a vector $\mathbf{b} \in \mathbb{F}^n$, and an index $j \in \{1, \dots, n\}$, we denote by $A_j(\mathbf{b})$ the matrix obtained from A by replacing the j-th column of A with $\mathbf{b}$. For example, if

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix},$$

then

- $A_1(\mathbf{b}) = \begin{bmatrix} 4 & 1 & 1 \\ 5 & 2 & 2 \\ 6 & 0 & 3 \end{bmatrix}$;
- $A_2(\mathbf{b}) = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 5 & 2 \\ 0 & 6 & 3 \end{bmatrix}$;
- $A_3(\mathbf{b}) = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 6 \end{bmatrix}$.

Cramer's rule. *Let A be an invertible matrix in $\mathbb{F}^{n \times n}$, and let $\mathbf{b} \in \mathbb{F}^n$. Then the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely*

$$\mathbf{x} = \begin{bmatrix} \frac{det(A_1(\mathbf{b}))}{det(A)} & \frac{det(A_2(\mathbf{b}))}{det(A)} & \dots & \frac{det(A_n(\mathbf{b}))}{det(A)} \end{bmatrix}^T.$$

Proof. Since A is invertible, we know that the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$. Now, for this solution $\mathbf{x}$, we set $\mathbf{x} = \left[\begin{array}{ccc} x_1 & \dots & x_n \end{array}\right]^T$. Our goal is to show that

$$\mathbf{x} \quad = \quad \left[\begin{array}{cccc} \frac{\det(A_1(\mathbf{b}))}{\det(A)} & \frac{\det(A_2(\mathbf{b}))}{\det(A)} & \dots & \frac{\det(A_n(\mathbf{b}))}{\det(A)} \end{array}\right]^T.$$

Fix an index $j \in \{1, \dots, n\}$. We must show that

$$x_j \quad = \quad \frac{\det(A_j(\mathbf{b}))}{\det(A)}.$$

Set $A = \left[\begin{array}{ccc} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{array}\right]$. Using the fact that $A\mathbf{x} = \mathbf{b}$ and the fact that $A\mathbf{x} = \sum_{i=1}^n x_i\mathbf{a}_i$ (by the definition of matrix-vector multiplication), we compute:

$$\begin{aligned}
\det(A_j(\mathbf{b})) \quad &= \quad \det\Big(\left[\begin{array}{cccccc} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{b} & \mathbf{a}_{j+1} & \dots \mathbf{a}_n \end{array}\right]\Big) \\
&= \quad \det\Big(\left[\begin{array}{cccccc} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & A\mathbf{x} & \mathbf{a}_{j+1} & \dots \mathbf{a}_n \end{array}\right]\Big) \\
&= \quad \det\Big(\left[\begin{array}{cccccc} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \sum\limits_{i=1}^n x_i\mathbf{a}_i & \mathbf{a}_{j+1} & \dots \mathbf{a}_n \end{array}\right]\Big) \\
&\overset{(*)}{=} \quad \sum_{i=1}^n x_i \det\Big(\left[\begin{array}{cccccc} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_i & \mathbf{a}_{j+1} & \dots \mathbf{a}_n \end{array}\right]\Big) \\
&\overset{(**)}{=} \quad x_j \det\Big(\left[\begin{array}{cccccc} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_j & \mathbf{a}_{j+1} & \dots \mathbf{a}_n \end{array}\right]\Big) \\
&= \quad x_j \det(A),
\end{aligned}$$

where (*) follows from Proposition 3.1(a) of Lecture Notes 15, and (**) follows from the fact that for all $i \in \{1, \dots, n\} \setminus \{j\}$, the matrix

$$\left[\begin{array}{cccccc} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_i & \mathbf{a}_{j+1} & \dots \mathbf{a}_n \end{array}\right]$$

has two identical columns and therefore (by Proposition 2.4 of Lecture Notes 15) has determinant zero. We have now shown that

$$\det(A_j(\mathbf{b})) \quad = \quad x_j \det(A).$$

Since A is invertible, Theorem 5.1 of Lecture Notes 15 guarantees that $\det(A) \neq 0$. So, we can divide both sides of the equality above by $\det(A)$ to obtain

$$x_j \quad = \quad \frac{\det(A_j(\mathbf{b}))}{\det(A)}.$$

This completes the argument. □

Example 3.1. *Let*

$$A \quad = \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{b} \quad = \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_3$*. Solve the matrix-vector equation* $A\mathbf{x} = \mathbf{b}$*.*

Solution. Note that $\det(A) = 2$, and in particular, A is invertible (by Theorem 5.1 from Lecture Notes 15). So, Cramer's rule applies. We compute:

- $\det(A_1(\mathbf{b})) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & 1 \end{vmatrix} = 2;$
- $\det(A_2(\mathbf{b})) = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 1;$
- $\det(A_3(\mathbf{b})) = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0.$

By Cramer's rule, $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely

$$\begin{aligned} \mathbf{x} \quad &= \quad \begin{bmatrix} \frac{\det(A_1(\mathbf{b}))}{\det(A)} & \frac{\det(A_2(\mathbf{b}))}{\det(A)} & \frac{\det(A_3(\mathbf{b}))}{\det(A)} \end{bmatrix}^T \\ &= \quad \begin{bmatrix} \frac{2}{2} & \frac{1}{2} & \frac{0}{2} \end{bmatrix}^T \\ &= \quad \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^T. \end{aligned}$$

□

4 The adjugate matrix

Given a matrix $A \in \mathbb{F}^{n \times n}$ (with $n \geq 2$), with cofactors $C_{i,j} = (-1)^{i+j}\det(A_{i,j})$ (for $i, j \in \{1, \dots, n\}$), the *cofactor matrix* of A is the matrix $[C_{i,j}]_{n \times n}$. The *adjugate matrix* (also called the *classical adjoint*) of A, denoted by $\text{adj}(A)$, is the transpose of the cofactor matrix of A, i.e.

$$\text{adj}(A) \quad := \quad [C_{i,j}]_{n \times n}{}^T.$$

So, the i, j-th entry of $\text{adj}(A)$ is the cofactor $C_{j,i}$ (note the swapping of the indices).

Example 4.1. *Consider the matrix*

$$A \quad = \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$*. Compute the cofactor and adjugate matrices of* A*.*

Solution. For all $i, j \in \{1, 2, 3\}$, we let $C_{i,j} = (-1)^{i+j}\det(A_{i,j})$. (So, the $C_{i,j}$'s are the cofactors of A.) We compute:

- $C_{1,1} = (-1)^{1+1} \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = 6;$
- $C_{1,2} = (-1)^{1+2} \begin{vmatrix} 0 & 2 \\ 0 & 3 \end{vmatrix} = 0;$
- $C_{1,3} = (-1)^{1+3} \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0;$
- $C_{2,1} = (-1)^{2+1} \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = -3;$
- $C_{2,2} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = 3;$
- $C_{2,3} = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0;$
- $C_{3,1} = (-1)^{3+1} \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0;$
- $C_{3,2} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2;$
- $C_{3,3} = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2.$

So, the cofactor matrix of A is

$$\begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} \quad = \quad \begin{bmatrix} 6 & 0 & 0 \\ -3 & 3 & 0 \\ 0 & -2 & 2 \end{bmatrix}.$$

The adjugate matrix of A is the transpose of the cofactor matrix, i.e.

$$\operatorname{adj}(A) \quad = \quad \begin{bmatrix} 6 & -3 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix}.$$

□

Theorem 4.2. *Let A be an invertible matrix in $\mathbb{F}^{n\times n}$. Then*

$$A^{-1} \quad = \quad \tfrac{1}{det(A)}\,adj(A).$$

Proof. Since A is invertible, Theorem 5.1 of Lecture Notes 15 guarantees that $\det(A) \neq 0$, and in particular, the expression $\frac{1}{\det(A)}\text{adj}(A)$ is defined. We will prove the theorem by showing that matrices A^{-1} and $\frac{1}{\det(A)}\text{adj}(A)$ have the same corresponding entries. Fix indices $i, j \in \{1, \dots, n\}$. By the definition of $\text{adj}(A)$, we see that the i, j-th entry of the matrix $\frac{1}{\det(A)}\text{adj}(A)$ is

$$\tfrac{(-1)^{j+i}\det(A_{j,i})}{\det(A)}.$$

We will use Cramer's rule to show that this is also the i, j-th entry of the matrix A^{-1}.

Set $A^{-1} = \begin{bmatrix} \mathbf{a}_1^* & \dots \mathbf{a}_n^* \end{bmatrix}$. Since $AA^{-1} = I_n$, we have that

$$A \begin{bmatrix} \mathbf{a}_1^* & \dots \mathbf{a}_n^* \end{bmatrix} \quad = \quad \begin{bmatrix} \mathbf{e}_1 & \dots & \mathbf{e}_n \end{bmatrix},$$

and consequently (by the definition of matrix-vector multiplication), that

$$\begin{bmatrix} A\mathbf{a}_1^* & \dots & A\mathbf{a}_n^* \end{bmatrix} \quad = \quad \begin{bmatrix} \mathbf{e}_1 & \dots & \mathbf{e}_n \end{bmatrix}.$$

In particular, the two matrices above have the same j-th column, and so $A\mathbf{a}_j^* = \mathbf{e}_j$, i.e. $\mathbf{a}_j^*$ is the solution of the equation $A\mathbf{x} = \mathbf{e}_j$ (this solution is unique because A is invertible). So, by Cramer's rule, we have that

$$\mathbf{a}_j^* \quad = \quad \begin{bmatrix} \frac{\det(A_1(\mathbf{e}_j))}{\det(A)} & \dots & \frac{\det(A_n(\mathbf{e}_j))}{\det(A)} \end{bmatrix}^T.$$

The i-th entry of $\mathbf{a}_j^*$ is

$$\tfrac{\det(A_i(\mathbf{e}_j))}{\det(A)}.$$

By Laplace explansion along the i-th column, we get that

$$\det(A_i(\mathbf{e}_j)) \quad = \quad (-1)^{j+i}\det(A_{j,i}).$$

So, the i-th entry of $\mathbf{a}_j^*$ (which is precisely the i, j-th entry of A^{-1}) is

$$\tfrac{(-1)^{j+i}\det(A_{j,i})}{\det(A)},$$

which is what we needed to show. □

Example 4.3. *Show that the matrix*

$$A \quad = \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix},$$

(with entries understood to be in $\mathbb{R}$) is invertible, and using Theorem 4.2, find its inverse A^{-1}.

Solution. The matrix A is upper triangular, and so its determinant can be computed by multiplying the entries along the main diagonal. So, $\det(A) = 1 \cdot 2 \cdot 3 = 6$. Since $\det(A) \neq 0$, Theorem 5.1 from Lecture Notes 15 guarantees that A is invertible. In Example 4.1, we compute the adjugate matrix of A:

$$\mathrm{adj}(A) = \begin{bmatrix} 6 & -3 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix}.$$

So, by Theorem 4.2, we have that

$$\begin{aligned} A^{-1} &= \tfrac{1}{\det(A)}\mathrm{adj}(A) \\ &= \tfrac{1}{6}\begin{bmatrix} 6 & -3 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & -1/3 \\ 0 & 0 & 1/3 \end{bmatrix}. \end{aligned}$$

□

Corollary 4.4. *Let $a, b, c, d \in \mathbb{F}$. Then the matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad \neq bc$, and in this case, the inverse of A is given by the formula

$$A^{-1} = \tfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Proof. By Theorem 5.1 from Lecture Notes 15, we know that A is invertible if and only if $\det(A) \neq 0$. Clearly, $\det(A) = ad - bc$, and it follows that A is invertible if and only if $ad - bc \neq 0$, i.e. if and only if $ad \neq bc$.

Now, assume that A is invertible, so that $ad \neq bc$. We first compute the cofactors $C_{i,j}$ of A:

- $C_{1,1} = (-1)^{1+1}\det(A_{1,1}) = d$;
- $C_{1,2} = (-1)^{1+2}\det(A_{1,2}) = -c$;
- $C_{2,1} = (-1)^{2+1}\det(A_{2,1}) = -b$;
- $C_{2,2} = (-1)^{2+2}\det(A_{2,2}) = a$.

The cofactor matrix of A is

$$\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.$$

The adjugate matrix of A is the transpose of the cofactor matrix, i.e.

$$\operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

By Theorem 4.2, we now have that

$$A^{-1} = \tfrac{1}{\det(A)}\operatorname{adj}(A) = \tfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

which is what we needed to show. □