# Linear Algebra 2

## Lecture #15
## Determinants

Irena Penev

March 16, 2023

- In this lecture, we begin our study of determinants.

- In this lecture, we begin our study of determinants.
- The definition of determinants involves permutations.
    - So, we begin by briefly reviewing permutations.
    - For a more detailed exposition of permutations, see section 3 of Lecture Notes 5.

- A *permutation* of a set $X$ is any bijection from $X$ to itself.

- A *permutation* of a set $X$ is any bijection from $X$ to itself.
- For a positive integer $n$, $S_n$ denotes the set of all permutations of the set $\{1, \ldots, n\}$.

- A *permutation* of a set $X$ is any bijection from $X$ to itself.
- For a positive integer $n$, $S_n$ denotes the set of all permutations of the set $\{1, \ldots, n\}$.
- $S_n$ is a group under the composition of functions.

- A *permutation* of a set $X$ is any bijection from $X$ to itself.
- For a positive integer $n$, $S_n$ denotes the set of all permutations of the set $\{1, \ldots, n\}$.
- $S_n$ is a group under the composition of functions.
- Any permutation in $S_n$ has a disjoint cycle decomposition.
  - For instance, we have the following permutation in $S_9$:

$$\begin{aligned} \pi &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 4 & 1 & 9 & 8 & 7 & 6 & 3 \end{pmatrix} \\ &= (125934)(68)(7). \end{aligned}$$

- A *permutation* of a set $X$ is any bijection from $X$ to itself.
- For a positive integer $n$, $S_n$ denotes the set of all permutations of the set $\{1, \ldots, n\}$.
- $S_n$ is a group under the composition of functions.
- Any permutation in $S_n$ has a disjoint cycle decomposition.
  - For instance, we have the following permutation in $S_9$:

$$\begin{aligned} \pi &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 4 & 1 & 9 & 8 & 7 & 6 & 3 \end{pmatrix} \\ &= (125934)(68)(7). \end{aligned}$$

- When $n$ is clear from context, cycles of length one may be omitted.

- A *permutation* of a set $X$ is any bijection from $X$ to itself.
- For a positive integer $n$, $S_n$ denotes the set of all permutations of the set $\{1, \ldots, n\}$.
- $S_n$ is a group under the composition of functions.
- Any permutation in $S_n$ has a disjoint cycle decomposition.
  - For instance, we have the following permutation in $S_9$:

$$\begin{aligned} \pi &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 4 & 1 & 9 & 8 & 7 & 6 & 3 \end{pmatrix} \\ &= (125934)(68)(7). \end{aligned}$$

- When $n$ is clear from context, cycles of length one may be omitted.
- The identity permutation in $S_n$ is denoted by 1, i.e. $1 = (1)(2)\ldots(n)$.

- A *permutation* of a set $X$ is any bijection from $X$ to itself.
- For a positive integer $n$, $S_n$ denotes the set of all permutations of the set $\{1, \ldots, n\}$.
- $S_n$ is a group under the composition of functions.
- Any permutation in $S_n$ has a disjoint cycle decomposition.
    - For instance, we have the following permutation in $S_9$:

$$\begin{aligned} \pi &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 4 & 1 & 9 & 8 & 7 & 6 & 3 \end{pmatrix} \\ &= (125934)(68)(7). \end{aligned}$$

- When $n$ is clear from context, cycles of length one may be omitted.
- The identity permutation in $S_n$ is denoted by 1, i.e. $1 = (1)(2)\ldots(n)$.
- A *transposition* in $S_n$ is a permutation of the form $(ij)$, for some distinct $i, j \in \{1, \ldots, n\}$.

- The *sign* of a permutation in $S_n$ is $(-1)^{n-k}$, where $k$ is the number of cycles in the disjoint cycle decomposition of that permutation (with cycles of length one included).

- The *sign* of a permutation in $S_n$ is $(-1)^{n-k}$, where $k$ is the number of cycles in the disjoint cycle decomposition of that permutation (with cycles of length one included).
  - For example, the sign of the permutation

$$\begin{aligned} \pi &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 4 & 1 & 9 & 8 & 7 & 6 & 3 \end{pmatrix} \\ &= (125934)(68)(7). \end{aligned}$$

is $\text{sgn}(\pi) = (-1)^{9-3} = 1$.

- The *sign* of a permutation in $S_n$ is $(-1)^{n-k}$, where $k$ is the number of cycles in the disjoint cycle decomposition of that permutation (with cycles of length one included).
  - For example, the sign of the permutation

$$\begin{aligned} \pi &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 4 & 1 & 9 & 8 & 7 & 6 & 3 \end{pmatrix} \\ &= (125934)(68)(7). \end{aligned}$$

  is $\text{sgn}(\pi) = (-1)^{9-3} = 1$.
- A permutation is *even* if its sign is 1, and it is *odd* if its sign is $-1$.

- The *sign* of a permutation in $S_n$ is $(-1)^{n-k}$, where $k$ is the number of cycles in the disjoint cycle decomposition of that permutation (with cycles of length one included).
  - For example, the sign of the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 4 & 1 & 9 & 8 & 7 & 6 & 3 \end{pmatrix}$$

$$= (125934)(68)(7).$$

  is $\text{sgn}(\pi) = (-1)^{9-3} = 1$.
- A permutation is *even* if its sign is 1, and it is *odd* if its sign is $-1$.
- The set of all even permutations in $S_n$ is denoted by $A_n$; this is a subgroup of $S_n$, and it is called the *alternating group* of degree $n$.

- The *sign* of a permutation in $S_n$ is $(-1)^{n-k}$, where $k$ is the number of cycles in the disjoint cycle decomposition of that permutation (with cycles of length one included).
  - For example, the sign of the permutation

$$\begin{aligned} \pi &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 4 & 1 & 9 & 8 & 7 & 6 & 3 \end{pmatrix} \\ &= (125934)(68)(7). \end{aligned}$$

    is $\operatorname{sgn}(\pi) = (-1)^{9-3} = 1$.
- A permutation is *even* if its sign is 1, and it is *odd* if its sign is $-1$.
- The set of all even permutations in $S_n$ is denoted by $A_n$; this is a subgroup of $S_n$, and it is called the *alternating group* of degree $n$.
- The sign of any transposition is $-1$, i.e. transpositions are odd.

### Proposition 3.2 from Lecture Notes 5

Let $n \geq 2$ be an integer, and let $\pi$ be a permutation in $S_n$. Then $\text{sgn}(\pi^{-1}) = \text{sgn}(\pi)$.

### Proposition 3.7 from Lecture Notes 5

Let $n \geq 2$ be an integer, and let $\sigma, \tau \in S_n$. Then $\text{sgn}(\sigma \circ \tau) = \text{sgn}(\sigma)\text{sgn}(\tau)$.

### Proposition 3.2 from Lecture Notes 5

Let $n \geq 2$ be an integer, and let $\pi$ be a permutation in $S_n$. Then $\text{sgn}(\pi^{-1}) = \text{sgn}(\pi)$.

### Proposition 3.7 from Lecture Notes 5

Let $n \geq 2$ be an integer, and let $\sigma, \tau \in S_n$. Then $\text{sgn}(\sigma \circ \tau) = \text{sgn}(\sigma)\text{sgn}(\tau)$.

- Since transpositions have sign $-1$, Proposition 3.7 from Lecture Notes 5 implies that

$$\text{sgn}(\sigma \circ \tau) \quad = \quad -\text{sgn}(\sigma)$$

for any permutation $\sigma$ and transposition $\tau$ in $S_n$.

- From now on, $\mathbb{F}$ is a fixed field.

- From now on, $\mathbb{F}$ is a fixed field.

**Definition**

The *determinant* of a matrix $A = [a_{i,j}]_{n\times n}$ in $\mathbb{F}^{n\times n}$, denoted by $\det(A)$ or $|A|$, is defined by

$$\begin{aligned}\det(A) &:= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \\ &= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.\end{aligned}$$

- From now on, $\mathbb{F}$ is a fixed field.

## Definition

The *determinant* of a matrix $A = [a_{i,j}]_{n\times n}$ in $\mathbb{F}^{n\times n}$, denoted by $\det(A)$ or $|A|$, is defined by

$$\begin{aligned}\det(A) &:= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \\ &= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.\end{aligned}$$

- Notation:

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{vmatrix} := \det\left( \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix} \right)$$

## Definition

The *determinant* of a matrix $A = [a_{i,j}]_{n \times n}$ in $\mathbb{F}^{n \times n}$, denoted by $\det(A)$ or $|A|$, is defined by

$$\begin{aligned} \det(A) &:= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}. \end{aligned}$$

## Definition

The *determinant* of a matrix $A = [a_{i,j}]_{n\times n}$ in $\mathbb{F}^{n\times n}$, denoted by $\det(A)$ or $|A|$, is defined by

$$\begin{aligned}\det(A) &:= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \\ &= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.\end{aligned}$$

- Only square matrices have determinants.

### Definition

The *determinant* of a matrix $A = [a_{i,j}]_{n\times n}$ in $\mathbb{F}^{n\times n}$, denoted by $\det(A)$ or $|A|$, is defined by

$$\begin{aligned}\det(A) &:= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \\ &= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.\end{aligned}$$

- Only square matrices have determinants.
- The determinant of a matrix in $\mathbb{F}^{n\times n}$ is always a scalar in $\mathbb{F}$.

### Definition

The *determinant* of a matrix $A = [a_{i,j}]_{n\times n}$ in $\mathbb{F}^{n\times n}$, denoted by $\det(A)$ or $|A|$, is defined by

$$\begin{aligned}\det(A) &:= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \\ &= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.\end{aligned}$$

- Only square matrices have determinants.
- The determinant of a matrix in $\mathbb{F}^{n\times n}$ is always a scalar in $\mathbb{F}$.
- Determinants have various interesting properties. For example:

### Theorem 5.1

Let $A \in \mathbb{F}^{n\times n}$. Then $A$ is invertible iff $\det(A) \neq 0$.

*Proof.* Later!

## Proposition 2.1

We have the following formulas for the determinants of $1 \times 1$, $2 \times 2$, and $3 \times 3$ matrices (with entries in the field $\mathbb{F}$):

(a) $$\begin{vmatrix} a_{1,1} \end{vmatrix} = a_{1,1};$$

(b) $$\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1};$$

(c) $$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} =$$
$$\begin{cases} \quad a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} \\ -a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3}. \end{cases}$$

## Proposition 2.1

We have the following formulas for the determinants of $1 \times 1$, $2 \times 2$, and $3 \times 3$ matrices (with entries in the field $\mathbb{F}$):

(a) $\begin{vmatrix} a_{1,1} \end{vmatrix} = a_{1,1};$

(b) $\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1};$

(c) $\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} =$
$$\begin{cases} \quad a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} \\ -a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3}. \end{cases}$$

*Proof.* We prove (b). The proof of (c) is similar, but a bit messier, and it can be found in the Lecture Notes. (The proof of (a) is easy.)

*Proof of (b).* WTS $\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$.

*Proof of (b).* WTS $\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$.

$S_2$ has two elements:

- $\sigma_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = (1)(2)$, with $\text{sgn}(\sigma_1) = 1$;
- $\sigma_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = (12)$, with $\text{sgn}(\sigma_2) = -1$.

*Proof of (b).* WTS $\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$.

$S_2$ has two elements:

- $\sigma_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = (1)(2)$, with $\text{sgn}(\sigma_1) = 1$;
- $\sigma_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = (12)$, with $\text{sgn}(\sigma_2) = -1$.

So, we have that

$$\begin{aligned} \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} &= \text{sgn}(\sigma_1)a_{1,\sigma_1(1)}a_{2,\sigma_1(2)} + \text{sgn}(\sigma_2)a_{1,\sigma_2(1)}a_{2,\sigma_2(2)} \\ &= a_{1,1}a_{2,2} - a_{1,2}a_{2,1}. \end{aligned}$$

## Proposition 2.1

We have the following formulas for the determinants of $1 \times 1$, $2 \times 2$, and $3 \times 3$ matrices (with entries in the field $\mathbb{F}$):

(a) $\begin{vmatrix} a_{1,1} \end{vmatrix} = a_{1,1}$;

(b) $\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$;

(c) $\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} =$

$$\begin{cases} \phantom{-}a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} \\ -a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3}. \end{cases}$$

## Proposition 2.1

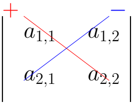
We have the following formulas for the determinants of $1 \times 1$, $2 \times 2$, and $3 \times 3$ matrices (with entries in the field $\mathbb{F}$):

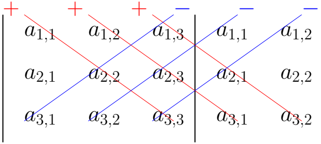
(a) $\begin{vmatrix} a_{1,1} \end{vmatrix} = a_{1,1}$;

(b) $\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$;

(c) $\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} =$

$$\begin{cases} \quad a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} \\ -a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3}. \end{cases}$$

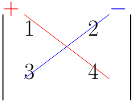


- For example, we can compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

in $\mathbb{R}^{2\times 2}$ by forming the diagram



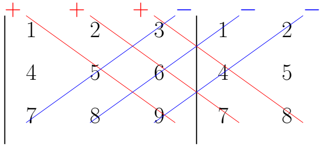

and the computing

$$\det(A) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2.$$

- Similarly, we can compute the determinant of the matrix

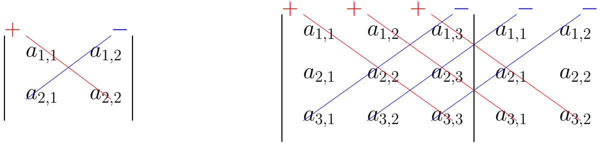
$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

in $\mathbb{R}^{3\times 3}$ by forming the diagram




and then computing

$$\begin{aligned} \det(B) &= \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \\ &= 1\cdot 5\cdot 9 + 2\cdot 6\cdot 7 + 3\cdot 4\cdot 8 - 3\cdot 5\cdot 7 - 1\cdot 6\cdot 8 - 2\cdot 4\cdot 9 \\ &= 0. \end{aligned}$$




- **Warning:** Do not try this with matrices of larger size!

### Theorem 2.2

For all $A \in \mathbb{F}^{n \times n}$, we have that $\det(A) = \det(A^T)$.

*Proof.*

## Theorem 2.2

For all $A \in \mathbb{F}^{n \times n}$, we have that $\det(A) = \det(A^T)$.

*Proof.* We set $A = [a_{i,j}]_{n \times n}$ and $A^T = [a^T_{i,j}]_{n \times n}$. So, for all $i, j \in \{1, \dots, n\}$, we have $a_{i,j} = a^T_{j,i}$.

## Theorem 2.2

For all $A \in \mathbb{F}^{n \times n}$, we have that $\det(A) = \det(A^T)$.

*Proof.* We set $A = [a_{i,j}]_{n \times n}$ and $A^T = [a^T_{i,j}]_{n \times n}$. So, for all $i, j \in \{1, \dots, n\}$, we have $a_{i,j} = a^T_{j,i}$. Now, we compute:

$$\begin{aligned}
\det(A^T) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a^T_{i,\sigma(i)} \\
&= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i} \\
&= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} a_{j,\sigma^{-1}(j)} \\
&\overset{\substack{\text{Prop. 3.2} \\ \text{(Lec. 5)}}}{=} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma^{-1}) \prod_{j=1}^{n} a_{j,\sigma^{-1}(j)} \\
&= \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{j=1}^{n} a_{j,\pi(j)} \\
&= \det(A)
\end{aligned}$$

Q.E.D.

- Some matrices whose determinants are zero

**Proposition 2.3**

Let $A = [a_{i,j}]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If $A$ has a zero row or a zero column, then $\det(A) = 0$.

**Proposition 2.4**

Let $A = [a_{i,j}]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If $A$ has two identical rows or two identical columns, then $\det(A) = 0$.

## Proposition 2.3

Let $A = [a_{i,j}]_{n\times n}$ be a matrix in $\mathbb{F}^{n\times n}$. If $A$ has a zero row or a zero column, then $\det(A) = 0$.

*Proof.*

### Proposition 2.3

Let $A = [a_{i,j}]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If $A$ has a zero row or a zero column, then $\det(A) = 0$.

*Proof.* In view of Theorem 2.2, it suffices to consider the case when $A$ has a zero row.

## Proposition 2.3

Let $A = [a_{i,j}]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If $A$ has a zero row or a zero column, then $\det(A) = 0$.

*Proof.* In view of Theorem 2.2, it suffices to consider the case when $A$ has a zero row. Suppose that that the $p$-th row of $A$ is a zero row.

## Proposition 2.3

Let $A = [a_{i,j}]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If $A$ has a zero row or a zero column, then $\det(A) = 0$.

*Proof.* In view of Theorem 2.2, it suffices to consider the case when $A$ has a zero row. Suppose that that the $p$-th row of $A$ is a zero row. Then for all $\sigma \in S_n$, we have that $a_{p,\sigma(p)} = 0$.

## Proposition 2.3

Let $A = [a_{i,j}]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If $A$ has a zero row or a zero column, then $\det(A) = 0$.

*Proof.* In view of Theorem 2.2, it suffices to consider the case when $A$ has a zero row. Suppose that that the $p$-th row of $A$ is a zero row. Then for all $\sigma \in S_n$, we have that $a_{p,\sigma(p)} = 0$. Consequently,

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{n,\sigma(n)} = 0,$$

which is what we needed to show. Q.E.D.

## Proposition 2.4

Let $A = [a_{i,j}]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If $A$ has two identical rows or two identical columns, then $\det(A) = 0$.

*Proof.*

### Proposition 2.4

Let $A = [a_{i,j}]_{n\times n}$ be a matrix in $\mathbb{F}^{n\times n}$. If $A$ has two identical rows or two identical columns, then $\det(A) = 0$.

*Proof.* In view of Theorem 2.2, it suffices to consider the case when $A$ has two identical rows.

### Proposition 2.4

Let $A = [a_{i,j}]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If $A$ has two identical rows or two identical columns, then $\det(A) = 0$.

*Proof.* In view of Theorem 2.2, it suffices to consider the case when $A$ has two identical rows. So, suppose that for some distinct $p, q \in \{1, \dots, n\}$, the $p$-th and $q$-th row of $A$ are the same. (In particular, $n \geq 2$.)

## Proposition 2.4

Let $A = [a_{i,j}]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If $A$ has two identical rows or two identical columns, then $\det(A) = 0$.

*Proof.* In view of Theorem 2.2, it suffices to consider the case when $A$ has two identical rows. So, suppose that for some distinct $p, q \in \{1, \dots, n\}$, the $p$-th and $q$-th row of $A$ are the same. (In particular, $n \geq 2$.) Now, let $A_n$ be the alternating group of degree $n$, i.e. the group of all even permutations in $S_n$, and let $O_n$ be the set of all odd permutations in $S_n$.

## Proposition 2.4

Let $A = [a_{i,j}]_{n\times n}$ be a matrix in $\mathbb{F}^{n\times n}$. If $A$ has two identical rows or two identical columns, then $\det(A) = 0$.

*Proof.* In view of Theorem 2.2, it suffices to consider the case when $A$ has two identical rows. So, suppose that for some distinct $p, q \in \{1, \ldots, n\}$, the $p$-th and $q$-th row of $A$ are the same. (In particular, $n \geq 2$.) Now, let $A_n$ be the alternating group of degree $n$, i.e. the group of all even permutations in $S_n$, and let $O_n$ be the set of all odd permutations in $S_n$. Obviously,

$$S_n = A_n \cup O_n \quad \text{and} \quad A_n \cap O_n = \emptyset.$$

## Proposition 2.4

Let $A = [a_{i,j}]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If $A$ has two identical rows or two identical columns, then $\det(A) = 0$.

*Proof.* In view of Theorem 2.2, it suffices to consider the case when $A$ has two identical rows. So, suppose that for some distinct $p, q \in \{1, \dots, n\}$, the $p$-th and $q$-th row of $A$ are the same. (In particular, $n \geq 2$.) Now, let $A_n$ be the alternating group of degree $n$, i.e. the group of all even permutations in $S_n$, and let $O_n$ be the set of all odd permutations in $S_n$. Obviously,

$$S_n = A_n \cup O_n \quad \text{and} \quad A_n \cap O_n = \emptyset.$$

Next, consider the transposition $\tau = (pq)$.

## Proposition 2.4

Let $A = [a_{i,j}]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If $A$ has two identical rows or two identical columns, then $\det(A) = 0$.

*Proof.* In view of Theorem 2.2, it suffices to consider the case when $A$ has two identical rows. So, suppose that for some distinct $p, q \in \{1, \dots, n\}$, the $p$-th and $q$-th row of $A$ are the same. (In particular, $n \geq 2$.) Now, let $A_n$ be the alternating group of degree $n$, i.e. the group of all even permutations in $S_n$, and let $O_n$ be the set of all odd permutations in $S_n$. Obviously,

$$S_n \;=\; A_n \cup O_n \qquad \text{and} \qquad A_n \cap O_n \;=\; \emptyset.$$

Next, consider the transposition $\tau = (pq)$. By Proposition 3.7 from Lecture Notes 5, for all $\sigma \in S_n$, we have that $\text{sgn}(\sigma \circ \tau) = -\text{sgn}(\sigma)$;

## Proposition 2.4

Let $A = [a_{i,j}]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If $A$ has two identical rows or two identical columns, then $\det(A) = 0$.

*Proof.* In view of Theorem 2.2, it suffices to consider the case when $A$ has two identical rows. So, suppose that for some distinct $p, q \in \{1, \dots, n\}$, the $p$-th and $q$-th row of $A$ are the same. (In particular, $n \geq 2$.) Now, let $A_n$ be the alternating group of degree $n$, i.e. the group of all even permutations in $S_n$, and let $O_n$ be the set of all odd permutations in $S_n$. Obviously,

$$S_n \;=\; A_n \cup O_n \qquad \text{and} \qquad A_n \cap O_n \;=\; \emptyset.$$

Next, consider the transposition $\tau = (pq)$. By Proposition 3.7 from Lecture Notes 5, for all $\sigma \in S_n$, we have that $\text{sgn}(\sigma \circ \tau) = -\text{sgn}(\sigma)$; it then readily follows that $O_n = \{\sigma \circ \tau \mid \sigma \in A_n\}$, and obviously, for all distinct $\sigma_1, \sigma_2 \in A_n$, we have that $\sigma_1 \circ \tau \neq \sigma_2 \circ \tau$.

## Proposition 2.4

Let $A = [a_{i,j}]_{n\times n}$ be a matrix in $\mathbb{F}^{n\times n}$. If $A$ has two identical rows or two identical columns, then $\det(A) = 0$.

*Proof (continued).* Reminder: $\tau = (pq)$.

***Claim.*** $\forall \sigma \in S_n : \prod_{i=1}^{n} a_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$.

## Proposition 2.4

Let $A = [a_{i,j}]_{n\times n}$ be a matrix in $\mathbb{F}^{n\times n}$. If $A$ has two identical rows or two identical columns, then $\det(A) = 0$.

*Proof (continued).* Reminder: $\tau = (pq)$.

**Claim.** $\forall \sigma \in S_n : \ \prod_{i=1}^{n} a_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$.

*Proof of the Claim.*

## Proposition 2.4

Let $A = [a_{i,j}]_{n\times n}$ be a matrix in $\mathbb{F}^{n\times n}$. If $A$ has two identical rows or two identical columns, then $\det(A) = 0$.

*Proof (continued).* Reminder: $\tau = (pq)$.

**Claim.** $\forall \sigma \in S_n: \ \prod_{i=1}^{n} a_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$.

*Proof of the Claim.* Fix $\sigma \in S_n$.

## Proposition 2.4

Let $A = [a_{i,j}]_{n\times n}$ be a matrix in $\mathbb{F}^{n\times n}$. If $A$ has two identical rows or two identical columns, then $\det(A) = 0$.

*Proof (continued).* Reminder: $\tau = (pq)$.

***Claim.*** $\forall \sigma \in S_n: \ \prod_{i=1}^{n} a_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$.

*Proof of the Claim.* Fix $\sigma \in S_n$. First, note that

- $a_{p,\sigma(p)} = a_{p,\sigma\circ\tau(q)} \overset{(*)}{=} a_{q,\sigma\circ\tau(q)}$,
- $a_{q,\sigma(q)} = a_{q,\sigma\circ\tau(p)} \overset{(*)}{=} a_{p,\sigma\circ\tau(p)}$,

where in both cases, (*) follows from the fact that the $p$-th and $q$-th row of $A$ are the same.

## Proposition 2.4

Let $A = [a_{i,j}]_{n\times n}$ be a matrix in $\mathbb{F}^{n\times n}$. If $A$ has two identical rows or two identical columns, then $\det(A) = 0$.

*Proof (continued).* Reminder: $\tau = (pq)$.

***Claim.*** $\forall \sigma \in S_n$ : $\prod_{i=1}^{n} a_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$.

*Proof of the Claim.* Fix $\sigma \in S_n$. First, note that

- $a_{p,\sigma(p)} = a_{p,\sigma\circ\tau(q)} \overset{(*)}{=} a_{q,\sigma\circ\tau(q)}$,
- $a_{q,\sigma(q)} = a_{q,\sigma\circ\tau(p)} \overset{(*)}{=} a_{p,\sigma\circ\tau(p)}$,

where in both cases, (*) follows from the fact that the $p$-th and $q$-th row of $A$ are the same. So, $a_{p,\sigma(p)}a_{q,\sigma(q)} = a_{p,\sigma\circ\tau(p)}a_{q,\sigma\circ\tau(q)}$.

## Proposition 2.4

Let $A = [a_{i,j}]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If $A$ has two identical rows or two identical columns, then $\det(A) = 0$.

*Proof (continued).* Reminder: $\tau = (pq)$.

***Claim.*** $\forall \sigma \in S_n$ : $\prod_{i=1}^{n} a_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma \circ \tau(i)}$.

*Proof of the Claim.* Fix $\sigma \in S_n$. First, note that

- $a_{p,\sigma(p)} = a_{p,\sigma \circ \tau(q)} \overset{(*)}{=} a_{q,\sigma \circ \tau(q)}$,
- $a_{q,\sigma(q)} = a_{q,\sigma \circ \tau(p)} \overset{(*)}{=} a_{p,\sigma \circ \tau(p)}$,

where in both cases, (*) follows from the fact that the $p$-th and $q$-th row of $A$ are the same. So, $a_{p,\sigma(p)} a_{q,\sigma(q)} = a_{p,\sigma \circ \tau(p)} a_{q,\sigma \circ \tau(q)}$.

On the other hand, $\forall i \in \{1, \dots, n\} \setminus \{p, q\}$: $a_{i,\sigma(i)} = a_{i,\sigma \circ \tau(i)}$.

## Proposition 2.4

Let $A = [a_{i,j}]_{n\times n}$ be a matrix in $\mathbb{F}^{n\times n}$. If $A$ has two identical rows or two identical columns, then $\det(A) = 0$.

*Proof (continued).* Reminder: $\tau = (pq)$.

**Claim.** $\forall \sigma \in S_n$ : $\prod_{i=1}^{n} a_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$.

*Proof of the Claim.* Fix $\sigma \in S_n$. First, note that

- $a_{p,\sigma(p)} = a_{p,\sigma\circ\tau(q)} \stackrel{(*)}{=} a_{q,\sigma\circ\tau(q)}$,
- $a_{q,\sigma(q)} = a_{q,\sigma\circ\tau(p)} \stackrel{(*)}{=} a_{p,\sigma\circ\tau(p)}$,

where in both cases, (*) follows from the fact that the $p$-th and $q$-th row of $A$ are the same. So, $a_{p,\sigma(p)}a_{q,\sigma(q)} = a_{p,\sigma\circ\tau(p)}a_{q,\sigma\circ\tau(q)}$.

On the other hand, $\forall i \in \{1,\dots,n\} \setminus \{p,q\}$: $a_{i,\sigma(i)} = a_{i,\sigma\circ\tau(i)}$.

It follows that $\prod_{i=1}^{n} a_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$. This proves the Claim.

We now compute:

$$\begin{aligned}
\det(A) &= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \\
&= \sum_{\sigma\in A_n} \underbrace{\operatorname{sgn}(\sigma)}_{=1} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} + \\
&\quad + \sum_{\pi\in O_n} \underbrace{\operatorname{sgn}(\pi)}_{=-1} a_{1,\pi(1)} \cdots a_{n,\pi(n)} \\
&= \sum_{\sigma\in A_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} - \sum_{\pi\in O_n} a_{1,\pi(1)} \cdots a_{n,\pi(n)} \\
&= \sum_{\sigma\in A_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} - \sum_{\sigma\in A_n} a_{1,\sigma\circ\tau(1)} \cdots a_{n,\sigma\circ\tau(n)} \\
&\overset{\text{Claim}}{=} \sum_{\sigma\in A_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} - \sum_{\sigma\in A_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \\
&= 0 \qquad \text{Q.E.D.}
\end{aligned}$$

- Some matrices whose determinants are zero

**Proposition 2.3**

Let $A = [a_{i,j}]_{n\times n}$ be a matrix in $\mathbb{F}^{n\times n}$. If $A$ has a zero row or a zero column, then $\det(A) = 0$.

**Proposition 2.4**

Let $A = [a_{i,j}]_{n\times n}$ be a matrix in $\mathbb{F}^{n\times n}$. If $A$ has two identical rows or two identical columns, then $\det(A) = 0$.

- In general, for matrices $A, B \in \mathbb{F}^{n \times n}$ and a scalar $\alpha \in \mathbb{F}$:

$$\det(A+B) \;\not\asymp\; \det(A)+\det(B) \quad \& \quad \det(\alpha A) \;\not\asymp\; \alpha\det(A).$$

- We do, however, have the following proposition (next slide).

## Proposition 3.1

Let $\mathbf{a}_1, \ldots, \mathbf{a}_{p-1}, \mathbf{a}_{p+1}, \ldots, \mathbf{a}_n \in \mathbb{F}^n$. Then:

(a) the function $f_{C_i} : \mathbb{F}^n \to \mathbb{F}$ given by

$$f_{C_i}(\mathbf{x}) = \det\left( \begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_{p-1} & \mathbf{x} & \mathbf{a}_{p+1} & \ldots & \mathbf{a}_n \end{bmatrix} \right)$$

for all $\mathbf{x} \in \mathbb{F}^n$ is linear;

(b) the function $f_{R_i} : \mathbb{F}^n \to \mathbb{F}$ given by

$$f_{R_i}(\mathbf{x}) = \det\left( \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_{p-1}^T \\ \mathbf{x}^T \\ \mathbf{a}_{p+1}^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} \right)$$

for all $\mathbf{x} \in \mathbb{F}^n$ is linear.

- Proof of Proposition 3.1: Lecture Notes.

- Proof of Proposition 3.1: Lecture Notes.

**Example 3.2**

- $\begin{vmatrix} 1 & \color{red}{2} & 1 \\ 2 & \color{red}{3} & 4 \\ 0 & \color{red}{1} & 5 \end{vmatrix} = \begin{vmatrix} 1 & \color{red}{1} & 1 \\ 2 & \color{red}{2} & 4 \\ 0 & \color{red}{-2} & 5 \end{vmatrix} + \begin{vmatrix} 1 & \color{red}{1} & 1 \\ 2 & \color{red}{1} & 4 \\ 0 & \color{red}{3} & 5 \end{vmatrix}$
- $\begin{vmatrix} \color{red}{3} & 2 & 4 \\ \color{red}{6} & -1 & 0 \\ \color{red}{-3} & 0 & 5 \end{vmatrix} = \color{red}{3} \begin{vmatrix} \color{red}{1} & 2 & 4 \\ \color{red}{2} & -1 & 0 \\ \color{red}{-1} & 0 & 5 \end{vmatrix}$
- $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ \color{red}{7} & \color{red}{3} & \color{red}{-2} \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ \color{red}{4} & \color{red}{4} & \color{red}{-2} \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ \color{red}{3} & \color{red}{-1} & \color{red}{0} \end{vmatrix}$
- $\begin{vmatrix} \color{red}{2} & \color{red}{-2} & \color{red}{4} \\ 1 & 0 & -2 \\ 2 & 1 & 4 \end{vmatrix} = \color{red}{2} \begin{vmatrix} \color{red}{1} & \color{red}{-1} & \color{red}{2} \\ 1 & 0 & -2 \\ 2 & 1 & 4 \end{vmatrix}$

- How do elementary row and column operations affect the value of the determinant?

- How do elementary row and column operations affect the value of the determinant?
- How can these operations be used to compute the determinant of a square matrix?

- How do elementary row and column operations affect the value of the determinant?
- How can these operations be used to compute the determinant of a square matrix?
- We begin by describing the endpoint of the calculation: triangular matrices.

- Given a square matrix $A = [a_{i,j}]_{n\times n}$ in $\mathbb{F}^{n\times n}$, we say that
  - $A$ is *upper triangular* if all entries of $A$ below the main diagonal are zero, i.e. if for all $i, j \in \{1, \dots, n\}$ s.t. $i > j$, we have that $a_{i,j} = 0$;
  - $A$ is *lower triangular* if all entries of $A$ above the main diagonal are zero, i.e. if for all $i, j \in \{1, \dots, n\}$ s.t. $i < j$, we have that $a_{i,j} = 0$;
  - $A$ is *triangular* if it is upper triangular or lower triangular.

$$\begin{bmatrix} * & * & * & \dots & * & * \\ 0 & * & * & \dots & * & * \\ 0 & 0 & * & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * \\ 0 & 0 & 0 & \dots & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & \dots & 0 & 0 \\ * & * & 0 & \dots & 0 & 0 \\ * & * & * & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \dots & * & 0 \\ * & * & * & \dots & * & * \end{bmatrix}$$

upper triangular matrix $\qquad$ lower triangular matrix

$$\begin{bmatrix} * & * & * & \dots & * & * \\ 0 & * & * & \dots & * & * \\ 0 & 0 & * & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * \\ 0 & 0 & 0 & \dots & 0 & * \end{bmatrix}$$

upper triangular matrix

$$\begin{bmatrix} * & 0 & 0 & \dots & 0 & 0 \\ * & * & 0 & \dots & 0 & 0 \\ * & * & * & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \dots & * & 0 \\ * & * & * & \dots & * & * \end{bmatrix}$$

lower triangular matrix

$$\begin{bmatrix} * & * & * & \dots & * & * \\ 0 & * & * & \dots & * & * \\ 0 & 0 & * & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * \\ 0 & 0 & 0 & \dots & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & \dots & 0 & 0 \\ * & * & 0 & \dots & 0 & 0 \\ * & * & * & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \dots & * & 0 \\ * & * & * & \dots & * & * \end{bmatrix}$$

upper triangular matrix — lower triangular matrix

- Any square matrix in row echelon form is in fact an upper triangular matrix.

$$\begin{bmatrix} * & * & * & \dots & * & * \\ 0 & * & * & \dots & * & * \\ 0 & 0 & * & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * \\ 0 & 0 & 0 & \dots & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & \dots & 0 & 0 \\ * & * & 0 & \dots & 0 & 0 \\ * & * & * & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \dots & * & 0 \\ * & * & * & \dots & * & * \end{bmatrix}$$

upper triangular matrix lower triangular matrix

- Any square matrix in row echelon form is in fact an upper triangular matrix. (However, not all upper triangular matrices are in row echelon form.)

$$\begin{bmatrix} * & * & * & \dots & * & * \\ 0 & * & * & \dots & * & * \\ 0 & 0 & * & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * \\ 0 & 0 & 0 & \dots & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & \dots & 0 & 0 \\ * & * & 0 & \dots & 0 & 0 \\ * & * & * & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \dots & * & 0 \\ * & * & * & \dots & * & * \end{bmatrix}$$

upper triangular matrix — lower triangular matrix

- Any square matrix in row echelon form is in fact an upper triangular matrix. (However, not all upper triangular matrices are in row echelon form.)
  - So, the row reduction algorithm performed on a square matrix will, in particular, yield an upper triangular matrix.

$$\begin{bmatrix} * & * & * & \dots & * & * \\ 0 & * & * & \dots & * & * \\ 0 & 0 & * & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * \\ 0 & 0 & 0 & \dots & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & \dots & 0 & 0 \\ * & * & 0 & \dots & 0 & 0 \\ * & * & * & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \dots & * & 0 \\ * & * & * & \dots & * & * \end{bmatrix}$$

upper triangular matrix — lower triangular matrix

- Any square matrix in row echelon form is in fact an upper triangular matrix. (However, not all upper triangular matrices are in row echelon form.)
  - So, the row reduction algorithm performed on a square matrix will, in particular, yield an upper triangular matrix.
- It turns out that the determinant of any triangular matrix is particularly easy to compute, as we now show.

**Proposition 4.1**

Let $A = [a_{i,j}]_{n\times n}$ be a triangular matrix in $\mathbb{F}^{n\times n}$. Then

$$\det(A) = \prod_{i=1}^{n} a_{i,i} = a_{1,1}a_{2,2}\dots a_{n,n},$$

that is, $\det(A)$ is equal to the product of entries on the main diagonal of $A$.

## Proposition 4.1

Let $A = [a_{i,j}]_{n \times n}$ be a triangular matrix in $\mathbb{F}^{n \times n}$. Then

$$\det(A) \quad = \quad \prod_{i=1}^{n} a_{i,i} \quad = \quad a_{1,1} a_{2,2} \ldots a_{n,n},$$

that is, $\det(A)$ is equal to the product of entries on the main diagonal of $A$.

- For example, we can compute the determinants of the following matrices in $\mathbb{R}^{3 \times 3}$:

  - $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 1 \cdot 4 \cdot 6 = 24;$
  - $\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 1 \cdot 3 \cdot 6 = 18.$

## Proposition 4.1

Let $A = [a_{i,j}]_{n \times n}$ be a triangular matrix in $\mathbb{F}^{n \times n}$. Then

$$\det(A) \quad = \quad \prod_{i=1}^{n} a_{i,i} \quad = \quad a_{1,1} a_{2,2} \dots a_{n,n},$$

that is, $\det(A)$ is equal to the product of entries on the main diagonal of $A$.

*Proof.*

### Proposition 4.1

Let $A = [a_{i,j}]_{n\times n}$ be a triangular matrix in $\mathbb{F}^{n\times n}$. Then

$$\det(A) \;=\; \prod_{i=1}^{n} a_{i,i} \;=\; a_{1,1}a_{2,2}\ldots a_{n,n},$$

that is, $\det(A)$ is equal to the product of entries on the main diagonal of $A$.

*Proof.* Note that the transpose of a lower triangular matrix is an upper triangular matrix, and moreover, the main diagonal remains unchanged when we take the transpose of a square matrix. So, in view of Theorem 2.2, it suffices to prove the result for the case when $A$ is upper triangular.

## Proposition 4.1

Let $A = [a_{i,j}]_{n\times n}$ be a triangular matrix in $\mathbb{F}^{n\times n}$. Then

$$\det(A) = \prod_{i=1}^{n} a_{i,i} = a_{1,1}a_{2,2}\dots a_{n,n},$$

that is, $\det(A)$ is equal to the product of entries on the main diagonal of $A$.

*Proof (continued).* Note that $\forall \sigma \in S_n \setminus \{1\}$, $\exists i \in \{1, \dots, n\}$ s.t. $i > \sigma(i)$,

## Proposition 4.1

Let $A = [a_{i,j}]_{n \times n}$ be a triangular matrix in $\mathbb{F}^{n \times n}$. Then

$$\det(A) \quad = \quad \prod_{i=1}^{n} a_{i,i} \quad = \quad a_{1,1}a_{2,2}\ldots a_{n,n},$$

that is, $\det(A)$ is equal to the product of entries on the main diagonal of $A$.

*Proof (continued).* Note that $\forall \sigma \in S_n \setminus \{1\}$, $\exists i \in \{1, \ldots, n\}$ s.t. $i > \sigma(i)$, and consequently, $a_{i,\sigma(i)} = 0$ (since $A$ is upper triangular).

## Proposition 4.1

Let $A = [a_{i,j}]_{n \times n}$ be a triangular matrix in $\mathbb{F}^{n \times n}$. Then

$$\det(A) \;=\; \prod_{i=1}^{n} a_{i,i} \;=\; a_{1,1}a_{2,2}\dots a_{n,n},$$

that is, $\det(A)$ is equal to the product of entries on the main diagonal of $A$.

*Proof (continued).* Note that $\forall \sigma \in S_n \setminus \{1\}$, $\exists i \in \{1, \dots, n\}$ s.t. $i > \sigma(i)$, and consequently, $a_{i,\sigma(i)} = 0$ (since $A$ is upper triangular). It follows that $\forall \sigma \in S_n \setminus \{1\}: \; a_{1,\sigma(1)}a_{2,\sigma(2)}\dots a_{n,\sigma(n)} = 0$,

## Proposition 4.1

Let $A = [a_{i,j}]_{n\times n}$ be a triangular matrix in $\mathbb{F}^{n\times n}$. Then

$$\det(A) = \prod_{i=1}^{n} a_{i,i} = a_{1,1}a_{2,2}\dots a_{n,n},$$

that is, $\det(A)$ is equal to the product of entries on the main diagonal of $A$.

*Proof (continued).* Note that $\forall \sigma \in S_n \setminus \{1\}$, $\exists i \in \{1,\dots,n\}$ s.t. $i > \sigma(i)$, and consequently, $a_{i,\sigma(i)} = 0$ (since $A$ is upper triangular). It follows that $\forall \sigma \in S_n \setminus \{1\}: \ a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)} = 0$, and so

$$\begin{aligned}\det(A) &= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)}\\ &= \operatorname{sgn}(1)a_{1,1}a_{2,2}\dots a_{n,n}\\ &= a_{1,1}a_{2,2}\dots a_{n,n}\end{aligned}$$

Q.E.D.

## Theorem 4.2

Let $A = [a_{i,j}]_{n\times n}$ be a matrix in $\mathbb{F}^{n\times n}$. Then all the following hold:

(a) if a matrix $B$ is obtained by swapping two rows or swapping two columns of $A$, then
$$\det(B) = -\det(A);$$

(b) if a matrix $B$ is obtained by multiplying some row or some column of $A$ by a scalar $\alpha \in \mathbb{F} \setminus \{0\}$, then
$$\det(B) = \alpha \det(A) \quad \text{and} \quad \det(A) = \alpha^{-1}\det(B);$$

(c) if a matrix $B$ is obtained from $A$ by adding a scalar multiple of one row (resp. column) of $A$ to another row (resp. column) of $A$, then
$$\det(B) = \det(A).$$

*Proof.* Later!

## Example 4.3

Compute the determinant of the matrix below (with entries understood to be in $\mathbb{R}$).

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{bmatrix}$$

*Solution.*

## Example 4.3

Compute the determinant of the matrix below (with entries understood to be in $\mathbb{R}$).

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{bmatrix}$$

*Solution.* We perform elementary row operations on $A$ (keeping track of the way that this changes the value of the determinant, as per Theorem 4.2) until we transform $A$ into a matrix in row echelon form. Square matrices in row echelon form are upper triangular, and so by Proposition 4.1, we can obtain their determinant by multiplying the entries on the main diagonal.

## Example 4.3

Compute the determinant of the matrix below (with entries understood to be in $\mathbb{R}$).

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{bmatrix}$$

*Solution.* We perform elementary row operations on $A$ (keeping track of the way that this changes the value of the determinant, as per Theorem 4.2) until we transform $A$ into a matrix in row echelon form. Square matrices in row echelon form are upper triangular, and so by Proposition 4.1, we can obtain their determinant by multiplying the entries on the main diagonal. We now compute (next slide):

*Solution (continued).*

$$
\begin{aligned}
\begin{vmatrix} 2 & 4 & 6 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{vmatrix}
&\overset{R_2 \to R_2 - R_1}{=} \begin{vmatrix} 2 & 4 & 6 \\ 0 & 0 & -2 \\ 3 & 3 & 7 \end{vmatrix} \\
&\overset{R_2 \leftrightarrow R_3}{=} -\begin{vmatrix} 2 & 4 & 6 \\ 3 & 3 & 7 \\ 0 & 0 & -2 \end{vmatrix} \\
&\overset{R_1 \to \frac{1}{2} R_1}{=} -2\begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 7 \\ 0 & 0 & -2 \end{vmatrix} \\
&\overset{R_2 \to R_1 - 3R_1}{=} -2\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & -2 \end{vmatrix} \\
&\overset{(*)}{=} (-2)1(-3)(-2) \\
&= -12
\end{aligned}
$$

## Example 4.4

Compute the determinant of the matrix below (with entries understood to be in $\mathbb{Z}_3$).

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 0 & 2 & 1 \\ 2 & 0 & 1 & 1 & 2 \\ 2 & 2 & 0 & 0 & 1 \\ 1 & 0 & 2 & 1 & 2 \end{bmatrix}$$

*Solution.*

## Example 4.4

Compute the determinant of the matrix below (with entries understood to be in $\mathbb{Z}_3$).

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 0 & 2 & 1 \\ 2 & 0 & 1 & 1 & 2 \\ 2 & 2 & 0 & 0 & 1 \\ 1 & 0 & 2 & 1 & 2 \end{bmatrix}$$

*Solution.* Here, we just notice that the second column is the sum of the first and third. This allows us to turn the second column into a zero column via two elementary column operations, which implies that $\det(A) = 0$. The detailed computation is as follows (next slide):

*Solution (continued).*

$$\begin{vmatrix} 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 0 & 2 & 1 \\ 2 & 0 & 1 & 1 & 2 \\ 2 & 2 & 0 & 0 & 1 \\ 1 & 0 & 2 & 1 & 2 \end{vmatrix} \underset{=}{C_2 \to C_2 - C_1} \begin{vmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 2 & 1 \\ 2 & 1 & 1 & 1 & 2 \\ 2 & 0 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 2 \end{vmatrix}$$

$$\underset{=}{C_2 \to C_2 - C_3} \begin{vmatrix} 1 & 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 2 & 1 \\ 2 & 0 & 1 & 1 & 2 \\ 2 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 1 & 2 \end{vmatrix}$$

$$\overset{(*)}{=} \quad 0,$$

where (*) follows from the fact that a matrix with a zero column has determinant zero (by Proposition 2.3).

**Theorem 5.1**

Let $A \in \mathbb{F}^{n \times n}$. Then $A$ is invertible iff $\det(A) \neq 0$.

*Proof.*

### Theorem 5.1

Let $A \in \mathbb{F}^{n \times n}$. Then $A$ is invertible iff $\det(A) \neq 0$.

*Proof.* We can transform $A$ into a matrix in reduced row echelon form via a sequence of elementary row operations.

**Theorem 5.1**

Let $A \in \mathbb{F}^{n \times n}$. Then $A$ is invertible iff $\det(A) \neq 0$.

*Proof.* We can transform $A$ into a matrix in reduced row echelon form via a sequence of elementary row operations. Each elementary row operation has the effect of multiplying the value of the determinant by some non-zero scalar.

### Theorem 5.1

Let $A \in \mathbb{F}^{n \times n}$. Then $A$ is invertible iff $\det(A) \neq 0$.

*Proof.* We can transform $A$ into a matrix in reduced row echelon form via a sequence of elementary row operations. Each elementary row operation has the effect of multiplying the value of the determinant by some non-zero scalar.

So, $\exists \alpha \in \mathbb{F} \setminus \{0\}$ s.t. $\det(A) = \alpha \det(\mathrm{RREF}(A))$.

### Theorem 5.1

Let $A \in \mathbb{F}^{n \times n}$. Then $A$ is invertible iff $\det(A) \neq 0$.

*Proof.* We can transform $A$ into a matrix in reduced row echelon form via a sequence of elementary row operations. Each elementary row operation has the effect of multiplying the value of the determinant by some non-zero scalar.

So, $\exists \alpha \in \mathbb{F} \setminus \{0\}$ s.t. $\det(A) = \alpha \det(\text{RREF}(A))$. Therefore, $\det(A) = 0$ iff $\det(\text{RREF}(A)) = 0$.

**Theorem 5.1**

Let $A \in \mathbb{F}^{n \times n}$. Then $A$ is invertible iff $\det(A) \neq 0$.

*Proof.* We can transform $A$ into a matrix in reduced row echelon form via a sequence of elementary row operations. Each elementary row operation has the effect of multiplying the value of the determinant by some non-zero scalar.

So, $\exists \alpha \in \mathbb{F} \setminus \{0\}$ s.t. $\det(A) = \alpha \det(\text{RREF}(A))$. Therefore, $\det(A) = 0$ iff $\det(\text{RREF}(A)) = 0$.

Moreover, RREF($A$) is an upper triangular matrix, and so its determinant is zero iff at least one entry on its main diagonal is zero.

## Theorem 5.1

Let $A \in \mathbb{F}^{n \times n}$. Then $A$ is invertible iff $\det(A) \neq 0$.

*Proof.* We can transform $A$ into a matrix in reduced row echelon form via a sequence of elementary row operations. Each elementary row operation has the effect of multiplying the value of the determinant by some non-zero scalar.

So, $\exists \alpha \in \mathbb{F} \setminus \{0\}$ s.t. $\det(A) = \alpha \det(\mathrm{RREF}(A))$. Therefore, $\det(A) = 0$ iff $\det(\mathrm{RREF}(A)) = 0$.

Moreover, $\mathrm{RREF}(A)$ is an upper triangular matrix, and so its determinant is zero iff at least one entry on its main diagonal is zero. We now have the following sequence of equivalences (next slide):

### Theorem 5.1

Let $A \in \mathbb{F}^{n\times n}$. Then $A$ is invertible iff $\det(A) \neq 0$.

*Proof (continued).*

$$\begin{aligned}
\det(A) = 0 \quad &\iff \quad \det(\mathrm{RREF}(A)) = 0 \\
&\iff \quad \text{at least one entry on the main diagonal of } \mathrm{RREF}(A) \text{ is } 0 \\
&\iff \quad \mathrm{RREF}(A) \neq I_n \\
&\iff \quad A \text{ is not invertible}
\end{aligned}$$

Q.E.D.

### Theorem 4.2

Let $A = [a_{i,j}]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. Then all the following hold:

(a) if a matrix $B$ is obtained by swapping two rows or swapping two columns of $A$, then

$$\det(B) \;=\; -\det(A);$$

(b) if a matrix $B$ is obtained by multiplying some row or some column of $A$ by a scalar $\alpha \in \mathbb{F} \setminus \{0\}$, then

$$\det(B) \;=\; \alpha \det(A) \quad \text{and} \quad \det(A) \;=\; \alpha^{-1} \det(B);$$

(c) if a matrix $B$ is obtained from $A$ by adding a scalar multiple of one row (resp. column) of $A$ to another row (resp. column) of $A$, then

$$\det(B) \;=\; \det(A).$$

*Proof.*

## Theorem 4.2

Let $A = [a_{i,j}]_{n\times n}$ be a matrix in $\mathbb{F}^{n\times n}$. Then all the following hold:

(a) if a matrix $B$ is obtained by swapping two rows or swapping two columns of $A$, then

$$\det(B) = -\det(A);$$

(b) if a matrix $B$ is obtained by multiplying some row or some column of $A$ by a scalar $\alpha \in \mathbb{F} \setminus \{0\}$, then

$$\det(B) = \alpha\det(A) \quad \text{and} \quad \det(A) = \alpha^{-1}\det(B);$$

(c) if a matrix $B$ is obtained from $A$ by adding a scalar multiple of one row (resp. column) of $A$ to another row (resp. column) of $A$, then

$$\det(B) = \det(A).$$

*Proof.* In view of Theorem 2.2, it suffices to prove the result for row operations only.

*Proof of (a).*

*Proof of (a).* Fix distinct indices $p, q \in \{1, \ldots, n\}$, and suppose that $B$ is obtained by swapping rows $p$ and $q$ of $A$ ("$R_p \leftrightarrow R_q$").

*Proof of (a).* Fix distinct indices $p, q \in \{1, \ldots, n\}$, and suppose that $B$ is obtained by swapping rows $p$ and $q$ of $A$ ("$R_p \leftrightarrow R_q$"). Set $B = [b_{i,j}]_{n \times n}$, so that

- $\forall j \in \{1, \ldots, n\}$: $b_{p,j} = a_{q,j}$ and $b_{q,j} = a_{p,j}$;
- $\forall i \in \{1, \ldots, n\} \setminus \{p, q\}$, $j \in \{1, \ldots, n\}$: $b_{i,j} = a_{i,j}$.

Next, consider the transposition $\tau = (pq)$ in $S_n$.

*Proof of (a).* Fix distinct indices $p, q \in \{1, \ldots, n\}$, and suppose that $B$ is obtained by swapping rows $p$ and $q$ of $A$ ("$R_p \leftrightarrow R_q$"). Set $B = [b_{i,j}]_{n\times n}$, so that

- $\forall j \in \{1, \ldots, n\}$: $b_{p,j} = a_{q,j}$ and $b_{q,j} = a_{p,j}$;
- $\forall i \in \{1, \ldots, n\} \setminus \{p, q\}$, $j \in \{1, \ldots, n\}$: $b_{i,j} = a_{i,j}$.

Next, consider the transposition $\tau = (pq)$ in $S_n$.

> ***Claim.*** $\forall \sigma \in S_n$: $\prod_{i=1}^{n} b_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$.

*Proof of the Claim.*

*Proof of (a).* Fix distinct indices $p, q \in \{1, \ldots, n\}$, and suppose that $B$ is obtained by swapping rows $p$ and $q$ of $A$ ("$R_p \leftrightarrow R_q$"). Set $B = [b_{i,j}]_{n\times n}$, so that

- $\forall j \in \{1, \ldots, n\}$: $b_{p,j} = a_{q,j}$ and $b_{q,j} = a_{p,j}$;
- $\forall i \in \{1, \ldots, n\} \setminus \{p, q\}$, $j \in \{1, \ldots, n\}$: $b_{i,j} = a_{i,j}$.

Next, consider the transposition $\tau = (pq)$ in $S_n$.

**Claim.** $\forall \sigma \in S_n$: $\prod_{i=1}^{n} b_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$.

*Proof of the Claim.* First, we note that

- $b_{p,\sigma(p)} = a_{q,\sigma(p)} = a_{q,\sigma\circ\tau(q)}$;
- $b_{q,\sigma(q)} = a_{p,\sigma(q)} = a_{p,\sigma\circ\tau(p)}$.

So, $b_{p,\sigma(p)} b_{q,\sigma(q)} = a_{p,\sigma\circ\tau(p)} a_{q,\sigma\circ\tau(q)}$.

*Proof of (a).* Fix distinct indices $p, q \in \{1, \dots, n\}$, and suppose that $B$ is obtained by swapping rows $p$ and $q$ of $A$ ("$R_p \leftrightarrow R_q$"). Set $B = [b_{i,j}]_{n \times n}$, so that

- $\forall j \in \{1, \dots, n\}$: $b_{p,j} = a_{q,j}$ and $b_{q,j} = a_{p,j}$;
- $\forall i \in \{1, \dots, n\} \setminus \{p, q\}$, $j \in \{1, \dots, n\}$: $b_{i,j} = a_{i,j}$.

Next, consider the transposition $\tau = (pq)$ in $S_n$.

***Claim.*** $\forall \sigma \in S_n$: $\prod_{i=1}^{n} b_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$.

*Proof of the Claim.* First, we note that

- $b_{p,\sigma(p)} = a_{q,\sigma(p)} = a_{q,\sigma\circ\tau(q)}$;
- $b_{q,\sigma(q)} = a_{p,\sigma(q)} = a_{p,\sigma\circ\tau(p)}$.

So, $b_{p,\sigma(p)} b_{q,\sigma(q)} = a_{p,\sigma\circ\tau(p)} a_{q,\sigma\circ\tau(q)}$.
On the other hand, $\forall i \in \{1, \dots, n\} \setminus \{p, q\}$: $b_{i,\sigma(i)} = a_{i,\sigma\circ\tau(i)}$.

*Proof of (a).* Fix distinct indices $p, q \in \{1, \dots, n\}$, and suppose that $B$ is obtained by swapping rows $p$ and $q$ of $A$ ("$R_p \leftrightarrow R_q$"). Set $B = [b_{i,j}]_{n \times n}$, so that

- $\forall j \in \{1, \dots, n\}$: $b_{p,j} = a_{q,j}$ and $b_{q,j} = a_{p,j}$;
- $\forall i \in \{1, \dots, n\} \setminus \{p, q\}$, $j \in \{1, \dots, n\}$: $b_{i,j} = a_{i,j}$.

Next, consider the transposition $\tau = (pq)$ in $S_n$.

**Claim.** $\forall \sigma \in S_n$: $\prod_{i=1}^{n} b_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$.

*Proof of the Claim.* First, we note that

- $b_{p,\sigma(p)} = a_{q,\sigma(p)} = a_{q,\sigma\circ\tau(q)}$;
- $b_{q,\sigma(q)} = a_{p,\sigma(q)} = a_{p,\sigma\circ\tau(p)}$.

So, $b_{p,\sigma(p)} b_{q,\sigma(q)} = a_{p,\sigma\circ\tau(p)} a_{q,\sigma\circ\tau(q)}$.
On the other hand, $\forall i \in \{1, \dots, n\} \setminus \{p, q\}$: $b_{i,\sigma(i)} = a_{i,\sigma\circ\tau(i)}$.
It follows that $\prod_{i=1}^{n} b_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$. This proves the Claim.

*Proof of (a) (continued).* **Claim.** $\forall \sigma \in S_n$: $\prod_{i=1}^{n} b_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$.

*Proof of (a) (continued).* **Claim.** $\forall \sigma \in S_n$: $\prod_{i=1}^{n} b_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$.

We now compute:

$$
\begin{aligned}
\det(B) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} b_{i,\sigma(i)} \\
&\overset{\text{Claim}}{=} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)} \\
&\overset{\substack{\text{Prop. 3.7} \\ \text{(Lec. 5)}}}{=} \sum_{\sigma \in S_n} \Big( -\operatorname{sgn}(\sigma \circ \tau) \Big) \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)} \\
&= -\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma \circ \tau) \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)} \\
&= -\sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{i=1}^{n} a_{i,\pi(i)} \\
&= -\det(A) \qquad \text{Q.E.D.}
\end{aligned}
$$

*Proof of (b).*

*Proof of (b).* Fix an index $p \in \{1, \dots, n\}$ and a scalar $\alpha \in \mathbb{F} \setminus \{0\}$, and suppose that $B$ is obtained by multiplying the $p$-th row of $A$ by $\alpha$ ("$R_p \to \alpha R_p$"). Set $B = [b_{i,j}]_{n \times n}$, so that

- $\forall j \in \{1, \dots, n\}$: $b_{p,j} = \alpha a_{p,j}$;
- $\forall i \in \{1, \dots, n\} \setminus \{p\}$, $j \in \{1, \dots, n\}$: $b_{i,j} = a_{i,j}$.

*Proof of (b).* Fix an index $p \in \{1, \dots, n\}$ and a scalar $\alpha \in \mathbb{F} \setminus \{0\}$, and suppose that $B$ is obtained by multiplying the $p$-th row of $A$ by $\alpha$ ("$R_p \to \alpha R_p$"). Set $B = [b_{i,j}]_{n \times n}$, so that

- $\forall j \in \{1, \dots, n\}$: $b_{p,j} = \alpha a_{p,j}$;
- $\forall i \in \{1, \dots, n\} \setminus \{p\}$, $j \in \{1, \dots, n\}$: $b_{i,j} = a_{i,j}$.

We now compute:

$$\begin{aligned}
& \det(B) \\
= \quad & \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) b_{1,\sigma(1)} \dots b_{n,\sigma(n)} \\
= \quad & \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{p-1,\sigma(p-1)} \Big( \alpha a_{p,\sigma(p)} \Big) a_{p+1,\sigma(p+1)} \dots a_{n,\sigma(n)} \\
= \quad & \alpha \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{n,\sigma(n)} \\
= \quad & \alpha \det(A).
\end{aligned}$$

*Proof of (b).* Fix an index $p \in \{1, \ldots, n\}$ and a scalar $\alpha \in \mathbb{F} \setminus \{0\}$, and suppose that $B$ is obtained by multiplying the $p$-th row of $A$ by $\alpha$ ("$R_p \to \alpha R_p$"). Set $B = [b_{i,j}]_{n \times n}$, so that

- $\forall j \in \{1, \ldots, n\}$: $b_{p,j} = \alpha a_{p,j}$;
- $\forall i \in \{1, \ldots, n\} \setminus \{p\}$, $j \in \{1, \ldots, n\}$: $b_{i,j} = a_{i,j}$.

We now compute:

$$
\begin{aligned}
& \det(B) \\
= \quad & \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) b_{1,\sigma(1)} \ldots b_{n,\sigma(n)} \\
= \quad & \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{1,\sigma(1)} \ldots a_{p-1,\sigma(p-1)} \Big( \alpha a_{p,\sigma(p)} \Big) a_{p+1,\sigma(p+1)} \ldots a_{n,\sigma(n)} \\
= \quad & \alpha \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{1,\sigma(1)} \ldots a_{n,\sigma(n)} \\
= \quad & \alpha \det(A).
\end{aligned}
$$

Since $\alpha \neq 0$, we deduce that $\det(A) = \alpha^{-1} \det(B)$. Q.E.D.

*Proof of (c).*

*Proof of (c).* Fix distinct indices $p, q \in \{1, \ldots, n\}$ and a scalar $\alpha \in \mathbb{F}$, and suppose that $B$ is obtained by adding $\alpha$ times row $p$ to row $q$ (“$R_q \to R_q + \alpha R_p$”). Set $B = [b_{i,j}]_{n \times n}$, so that

- $\forall j \in \{1, \ldots, n\}$: $b_{q,j} = a_{q,j} + \alpha a_{p,j}$;
- $\forall i \in \{1, \ldots, n\} \setminus \{q\}$, $j \in \{1, \ldots, n\}$: $b_{i,j} = a_{i,j}$.

*Proof of (c).* Fix distinct indices $p, q \in \{1, \ldots, n\}$ and a scalar $\alpha \in \mathbb{F}$, and suppose that $B$ is obtained by adding $\alpha$ times row $p$ to row $q$ ("$R_q \to R_q + \alpha R_p$"). Set $B = [b_{i,j}]_{n \times n}$, so that

- $\forall j \in \{1, \ldots, n\}$: $b_{q,j} = a_{q,j} + \alpha a_{p,j}$;
- $\forall i \in \{1, \ldots, n\} \setminus \{q\}$, $j \in \{1, \ldots, n\}$: $b_{i,j} = a_{i,j}$.

We then compute (next slide):

*Proof of (c) (continued).*

$$\det(B) = \begin{vmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{q-1,1} & \cdots & a_{q-1,n} \\ a_{q,1} + \alpha a_{p,1} & \cdots & a_{q,n} + \alpha a_{p,n} \\ a_{q+1,1} & \cdots & a_{q+1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{vmatrix}$$

$$\overset{(*)}{=} \underbrace{\begin{vmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{q-1,1} & \cdots & a_{q-1,n} \\ a_{q,1} & \cdots & a_{q,n} \\ a_{q+1,1} & \cdots & a_{q+1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{vmatrix}}_{=\det(A)} + \alpha \underbrace{\begin{vmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{q-1,1} & \cdots & a_{q-1,n} \\ a_{p,1} & \cdots & a_{p,n} \\ a_{q+1,1} & \cdots & a_{q+1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{vmatrix}}_{\overset{(**)}{=} 0} = \det(A),$$

where (*) follows from the fact that the determinant is linear in the $q$-th row (by Proposition 3.1), and (**) follows from the fact that a matrix with two identical rows (in this case, the $p$-th and $q$-th row) has determinant zero (by Proposition 2.4). Q.E.D.