Linear Algebra 2

Lecture #14
Orthogonal matrices

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March 9, 2023

- Some preliminaries

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Proposition 1.1

Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n\times n}$. If $AB = I_n$, then A and B are both invertible and are each other's inverses.

Proof. Problem 2(b) of HW#9 from Linear Algebra 1 (winter 2022).

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Proposition 1.2

Let V be a finite-dimensional vector space over $\mathbb{R}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$. Set $n := \dim(V)$, and let $\mathbf{u}_1, \dots, \mathbf{u}_n \in V$. Then $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is an orthonormal set in V if and only if $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is an orthonormal basis of V.

Proof.

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Proof. "$\Longleftarrow$" is obvious. For "$\Longrightarrow$," note that if $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is an orthonormal set, then it is linearly independent, and so since $\dim(V) = n$, it is a basis of V.

Definition

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Theorem 2.1

Let $Q \in \mathbb{R}^{n\times n}$. Then the following are equivalent:

(a) Q is orthogonal (i.e. satisfies $Q^T Q = I_n$);
(b) Q is invertible and satisfies $Q^{-1} = Q^T$;
(c) $QQ^T = I_n$;
(d) Q^T is orthogonal;
(e) Q is invertible and Q^{-1} is orthogonal;
(f) the columns of Q form an orthonormal basis of $\mathbb{R}^n$;
(g) the columns of Q^T form an orthonormal basis of $\mathbb{R}^n$.

- First some examples, then a proof.

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$$H(\mathbf{a}) \quad := \quad I_n - \tfrac{2}{\mathbf{a}^T\mathbf{a}}\mathbf{a}\mathbf{a}^T \quad = \quad I_n - \tfrac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T.$$

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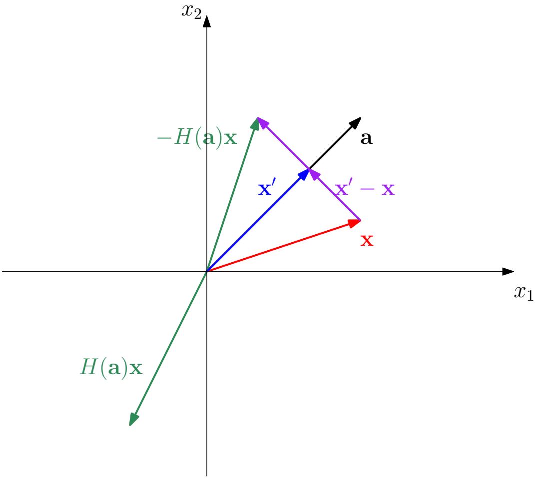
To see that $H(\mathbf{a})$ really is an orthogonal matrix, we perform the following simple calculation:

$$\begin{aligned}
H(\mathbf{a})^T\,H(\mathbf{a}) &= (I_n - \tfrac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T)^T\,(I_n - \tfrac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T) \\
&= (I_n^T - \tfrac{2}{\mathbf{a}\cdot\mathbf{a}}(\mathbf{a}\mathbf{a}^T)^T)\,(I_n - \tfrac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T) \\
&= (I_n - \tfrac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T)\,(I_n - \tfrac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T) \\
&= I_n - \tfrac{4}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T + \tfrac{4}{(\mathbf{a}\cdot\mathbf{a})^2}\mathbf{a}\,\underbrace{\mathbf{a}^T\mathbf{a}}_{=\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a}^T \\
&= I_n - \tfrac{4}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T + \tfrac{4}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T \\
&= I_n.
\end{aligned}$$

- Reminder: $H(\mathbf{a}) := I_n - \frac{2}{\mathbf{a}^T\mathbf{a}}\mathbf{a}\mathbf{a}^T = I_n - \frac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T$.



The Householder matrix $H(\mathbf{a})$ itself is the standard matrix of the linear operation that first reflects about the Span($\mathbf{a}$) line and then reflects about the origin.

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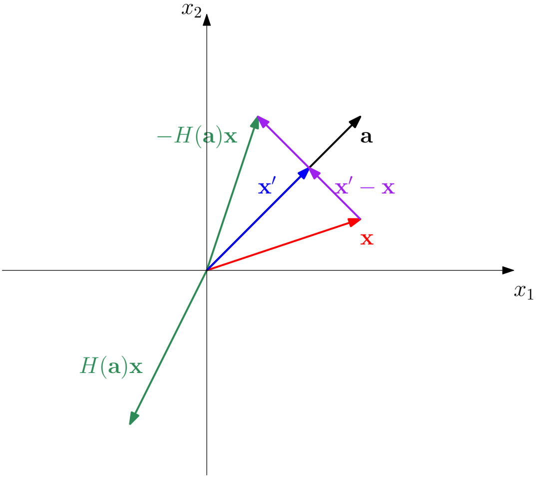

The Householder matrix $H(\mathbf{a})$ itself is the standard matrix of the linear operation that first reflects about the Span($\mathbf{a}$) line and then reflects about the origin. Let's check!

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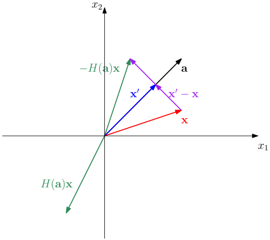

  - The standard matrix of orthogonal projection onto the line Span($\mathbf{a}$) is $\mathbf{a}(\mathbf{a}^T\mathbf{a})^{-1}\mathbf{a}^T = \mathbf{a}(\mathbf{a}\cdot\mathbf{a})^{-1}\mathbf{a}^T = \frac{1}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T$.
  - If $\mathbf{x}$ is any vector in $\mathbb{R}^n$ and $\mathbf{x}'$ is its projection onto Span($\mathbf{a}$), then the reflection of $\mathbf{x}$ about the line Span($\mathbf{a}$) is

$$\begin{aligned}
\mathbf{x} + 2(\mathbf{x}' - \mathbf{x}) &= 2\mathbf{x}' - \mathbf{x} \\
&= \frac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T\mathbf{x} - I_n\mathbf{x} \\
&= (\frac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T - I_n)\mathbf{x} \\
&= -H(\mathbf{a})\mathbf{x}.
\end{aligned}$$

  - $H(\mathbf{a})\mathbf{x}$ is the reflection of $-H(\mathbf{a})\mathbf{x}$ about the origin.

- Given an integer $n \geq 2$, indices $i, j \in \{1, \ldots, n\}$ such that $i < j$, and real numbers c and s such that $c^2 + s^2 = 1$, we define the *Givens matrix* $G_{i,j}(c, s)$ as follows:

$$
G_{i,j}(c,s) = \begin{array}{c} \\ \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \\ \end{array}
\overset{\begin{array}{ccccccccc} & & & i & & & j & & \end{array}}{
\begin{bmatrix}
1 & 0 & \dots 0 & 0 & 0 & \dots 0 & 0 & 0 & \dots 0 \\
0 & 1 & \dots 0 & 0 & 0 & \dots 0 & 0 & 0 & \dots 0 \\
\vdots & \vdots & \ddots \vdots & \vdots & \vdots & \ddots \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots 1 & 0 & 0 & \dots 0 & 0 & 0 & \dots 0 \\
0 & 0 & \dots 0 & c & 0 & \dots 0 & -s & 0 & \dots 0 \\
0 & 0 & \dots 0 & 0 & 1 & \dots 0 & 0 & 0 & \dots 0 \\
\vdots & \vdots & \dots \vdots & \vdots & \vdots & \ddots \vdots & \vdots & \vdots & 0 \\
0 & 0 & \dots 0 & 0 & 0 & \dots 1 & 0 & 0 & \dots 0 \\
0 & 0 & \dots 0 & s & 0 & \dots 0 & c & 0 & \dots 0 \\
0 & 0 & \dots 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \vdots \\
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\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\
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0 & 0 & \ldots & 0 & 0 & 0 & & 0 & 0 & 0 & \ldots & 1
\end{array}
\right]
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- $G_{i,j}(c, s)$ is orthogonal because its columns form an orthonormal basis of $\mathbb{R}^n$.

- Since $c^2 + s^2 = 1$, we see that there exists a real number (angle in radians) θ such that $c = \cos\theta$ and $s = \sin\theta$.

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- With this set-up, we see that $G_{i,j}(c, s)$ represents rotation by angle θ in the $x_i x_j$-plane.
- This is particularly easy to see in the case when $n = 2$. In that case, we have that

$$G_{1,2}(c, s) = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

which is precisely the standard matrix of counterclockwise rotation by angle θ.

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Proof. By Proposition 1.1, we have that (a), (b), and (c) are equivalent.

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Proof. By Proposition 1.1, we have that (a), (b), and (c) are equivalent. Moreover, since $(Q^T)^T = Q$, we have that (c) and (d) are equivalent. This proves that (a), (b), (c), and (d) are equivalent.

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Proof (continued). Next, (b) and (d) together imply (e).

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So far, we have established that (a), (b), (c), (d), and (e) are equivalent.

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$$Q^T Q = \begin{bmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \\ \vdots \\ \mathbf{q}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \dots & \mathbf{q}_n \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{q}_1 \cdot \mathbf{q}_1 & \mathbf{q}_1 \cdot \mathbf{q}_2 & \dots & \mathbf{q}_1 \cdot \mathbf{q}_n \\ \mathbf{q}_2 \cdot \mathbf{q}_1 & \mathbf{q}_2 \cdot \mathbf{q}_2 & \dots & \mathbf{q}_2 \cdot \mathbf{q}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{q}_n \cdot \mathbf{q}_1 & \mathbf{q}_n \cdot \mathbf{q}_2 & \dots & \mathbf{q}_n \cdot \mathbf{q}_n \end{bmatrix}.$$

So, $Q^T Q = I_n$ if and only if $\{\mathbf{q}_1, \dots, \mathbf{q}_n\}$ is an orthonormal set. In view of Proposition 1.2, it now follows that (a) and (f) are equivalent. Analogously, (d) and (g) are equivalent. Q.E.D.

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- Making new orthogonal matrices out of old ones:

Proposition 2.2

If $Q_1, Q_2 \in \mathbb{R}^{n\times n}$ are orthogonal, then so is their product $Q_1 Q_2$.

Proposition 2.3

Let $Q_1 \in \mathbb{R}^{m\times m}$ and $Q_2 \in \mathbb{R}^{n\times n}$ be orthogonal matrices. Then

$$Q = \left[\begin{array}{c|c} Q_1 & O_{m\times n} \\ \hline O_{n\times m} & Q_2 \end{array}\right]$$

is an orthogonal matrix in $\mathbb{R}^{(m+n)\times(m+n)}$.

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Proof. Assume $Q_1, Q_2 \in \mathbb{R}^{n\times n}$ are orthogonal. Then $Q_1^T Q_1 = I_n$ and $Q_2^T Q_2 = I_n$, and consequently,

$$(Q_1 Q_2)^T (Q_1 Q_2) = Q_2^T \underbrace{Q_1^T Q_1}_{=I_n} Q_2 = Q_2^T Q_2 = I_n.$$

So, $Q_1 Q_2$ is indeed orthogonal.

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Let $Q_1 \in \mathbb{R}^{m\times m}$ and $Q_2 \in \mathbb{R}^{n\times n}$ be orthogonal matrices. Then

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is an orthogonal matrix in $\mathbb{R}^{(m+n)\times(m+n)}$.

Proof. By hypothesis, we have that $Q_1^T Q_1 = I_m$ and $Q_2^T Q_2 = I_n$. We now compute:

$$\begin{aligned}
Q^T Q &= \left[\begin{array}{c|c} Q_1^T & O_{m\times n} \\ \hline O_{n\times m} & Q_2^T \end{array}\right] \left[\begin{array}{c|c} Q_1 & O_{m\times n} \\ \hline O_{n\times m} & Q_2 \end{array}\right] \\
&= \left[\begin{array}{c|c} Q_1^T Q_1 + O_{m\times n} O_{n\times m} & Q_1^T O_{m\times n} + O_{m\times n} Q_2 \\ \hline O_{n\times m} Q_1 + Q_2^T O_{n\times m} & O_{n\times m} O_{m\times n} + Q_2^T Q_2 \end{array}\right] \\
&= \left[\begin{array}{c|c} Q_1^T Q_1 & O_{m\times n} \\ \hline O_{n\times m} & Q_2^T Q_2 \end{array}\right] \\
&= \left[\begin{array}{c|c} I_m & O_{m\times n} \\ \hline O_{n\times m} & I_n \end{array}\right] \\
&= I_{m+n}.
\end{aligned}$$

So, Q is indeed an orthogonal matrix. Q.E.D.

Theorem 2.4

Let $Q = [q_{i,j}]_{n \times n}$ be an orthogonal matrix in $\mathbb{R}^{n \times n}$. Then all the following hold:

1. for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $(Q\mathbf{x}) \cdot (Q\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$;
2. for all $\mathbf{x} \in \mathbb{R}^n$, $||Q\mathbf{x}|| = ||\mathbf{x}||$;
3. for all $i, j \in \{1, \dots, n\}$, $|q_{i,j}| \leq 1$.

Proof.

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Proof. (a) For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have that

$$(Q\mathbf{x}) \cdot (Q\mathbf{y}) \;=\; (Q\mathbf{x})^T (Q\mathbf{x}) \;=\; \mathbf{x}^T \underbrace{Q^T Q}_{=I_n} \mathbf{y} \;=\; \mathbf{x}^T \mathbf{y} \;=\; \mathbf{x} \cdot \mathbf{y}.$$

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Proof. (a) For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have that

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(b) For $\mathbf{x} \in \mathbb{R}^n$, we have that

$$||Q\mathbf{x}|| \;=\; \sqrt{(Q\mathbf{x}) \cdot (Q\mathbf{x})} \;\overset{(a)}{=}\; \sqrt{\mathbf{x} \cdot \mathbf{x}} \;=\; ||\mathbf{x}||.$$

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Proof (continued).

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Proof (continued). (c) By Theorem 2.1, the columns of Q form an orthonormal basis. In particular, all columns of Q are of length one, and it follows that all entries of Q have absolute value at most 1.

Theorem 2.4

Let $Q = [q_{i,j}]_{n\times n}$ be an orthogonal matrix in $\mathbb{R}^{n\times n}$. Then all the following hold:

a. for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $(Q\mathbf{x}) \cdot (Q\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$;
b. for all $\mathbf{x} \in \mathbb{R}^n$, $||Q\mathbf{x}|| = ||\mathbf{x}||$;
c. for all $i, j \in \{1, \ldots, n\}$, $|q_{i,j}| \leq 1$.

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- Suppose that $Q \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, and that $f_Q : \mathbb{R}^n \to \mathbb{R}^n$ is given by $f_Q(\mathbf{x}) = Q\mathbf{x}$.
  - So, $f_Q : \mathbb{R}^n \to \mathbb{R}^n$ is the linear transformation whose standard matrix is Q.

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    - So, $f_Q : \mathbb{R}^n \to \mathbb{R}^n$ is the linear transformation whose standard matrix is Q.
- By Theorem 2.4(b), f_Q preserves vector length.
- On the other hand, recall that for non-zero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have that $\mathbf{x} \cdot \mathbf{y} = ||\mathbf{x}|| \, ||\mathbf{y}|| \cos\theta$, where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. So, Theorem 2.4(a) effectively states that f_Q preserves angles between vectors.

- Scalar product, coordinate vectors, and matrices of linear transformations

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Proposition 3.1

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with the scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$, and let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an orthonormal basis of V. Let $\cdot$ be the standard scalar product in $\mathbb{R}^n$ or $\mathbb{C}^n$. Then for all $\mathbf{x}, \mathbf{y} \in V$, we have that

$$\langle \mathbf{x}, \mathbf{y} \rangle \quad = \quad [\mathbf{x}]_{\mathcal{B}} \cdot [\mathbf{y}]_{\mathcal{B}}.$$

Proof.

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Proof. We prove the result for the case when V is a vector space over $\mathbb{C}$. The proof for $\mathbb{R}$ is similar but easier (because there are no complex conjugates).

- Scalar product, coordinate vectors, and matrices of linear transformations

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Proof. We prove the result for the case when V is a vector space over $\mathbb{C}$. The proof for $\mathbb{R}$ is similar but easier (because there are no complex conjugates). Fix $\mathbf{x}, \mathbf{y} \in V$. Since $\mathcal{B}$ is an orthonormal basis for V, we see that

$$\mathbf{x} = \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{u}_i \rangle \, \mathbf{u}_i \quad \text{and} \quad \mathbf{y} = \sum_{i=1}^{n} \langle \mathbf{y}, \mathbf{u}_i \rangle \, \mathbf{u}_i,$$

and consequently (next slide):

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Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with the scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$, and let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an orthonormal basis of V. Let $\cdot$ be the standard scalar product in $\mathbb{R}^n$ or $\mathbb{C}^n$. Then for all $\mathbf{x}, \mathbf{y} \in V$, we have that

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Proof (continued).

$$[\mathbf{x}]_{\mathcal{B}} \quad = \quad \begin{bmatrix} \langle \mathbf{x}, \mathbf{u}_1 \rangle \\ \vdots \\ \langle \mathbf{x}, \mathbf{u}_n \rangle \end{bmatrix} \qquad \text{and} \qquad [\mathbf{y}]_{\mathcal{B}} \quad = \quad \begin{bmatrix} \langle \mathbf{y}, \mathbf{u}_1 \rangle \\ \vdots \\ \langle \mathbf{y}, \mathbf{u}_n \rangle \end{bmatrix}.$$

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We now compute (next slide):

Proof (continued).

$$
\begin{aligned}
\langle \mathbf{x}, \mathbf{y} \rangle &= \Big\langle \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{u}_i \rangle\ \mathbf{u}_i, \sum_{j=1}^{n} \langle \mathbf{y}, \mathbf{u}_j \rangle\ \mathbf{u}_j \Big\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \Big\langle \langle \mathbf{x}, \mathbf{u}_i \rangle\ \mathbf{u}_i, \langle \mathbf{y}, \mathbf{u}_j \rangle\ \mathbf{u}_j \Big\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{u}_i \rangle \overline{\langle \mathbf{y}, \mathbf{u}_j \rangle}\ \langle \mathbf{u}_i, \mathbf{u}_j \rangle \\
&= \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{u}_i \rangle \overline{\langle \mathbf{y}, \mathbf{u}_i \rangle} && \text{by orthonormality} \\
&= \begin{bmatrix} \langle \mathbf{x}, \mathbf{u}_1 \rangle \\ \vdots \\ \langle \mathbf{x}, \mathbf{u}_n \rangle \end{bmatrix} \cdot \begin{bmatrix} \langle \mathbf{y}, \mathbf{u}_1 \rangle \\ \vdots \\ \langle \mathbf{y}, \mathbf{u}_n \rangle \end{bmatrix} \\
&= [\mathbf{x}]_{\mathcal{B}} \cdot [\mathbf{y}]_{\mathcal{B}} && \text{Q.E.D.}
\end{aligned}
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Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with the scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$, and let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an orthonormal basis of V. Let $\cdot$ be the standard scalar product in $\mathbb{R}^n$ or $\mathbb{C}^n$. Then for all $\mathbf{x}, \mathbf{y} \in V$, we have that

$$\langle \mathbf{x}, \mathbf{y} \rangle = [\mathbf{x}]_{\mathcal{B}} \cdot [\mathbf{y}]_{\mathcal{B}}.$$

Theorem 3.2

Let U and V be non-trivial, finite-dimensional vector spaces over $\mathbb{R}$. Assume that U is equipped with a scalar product $\langle \cdot, \cdot \rangle_U$ and the induced norm $|| \cdot ||_U$, and that V is equipped with a scalar product $\langle \cdot, \cdot \rangle_V$ and the induced norm $|| \cdot ||_V$. Let $\mathcal{B}_U = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ and $\mathcal{B}_V = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be orthonormal bases for U and V, respectively, and let $f : U \to V$ be a linear transformation. Then the following two statements are equivalent:

(i) the columns of the $n \times m$ matrix ${}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}$ form an orthonormal set of vectors in $\mathbb{R}^n$;[a]

(ii) for all $\mathbf{x}, \mathbf{y} \in U$, we have that $\langle f(\mathbf{x}), f(\mathbf{y}) \rangle_V = \langle \mathbf{x}, \mathbf{y} \rangle_U$.

---

[a] However, despite Theorem 2.1, this does not necessarily mean that the matrix ${}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}$ is orthogonal. This is because ${}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}$ is an $n \times m$ matrix, and it is possible that $m \neq n$, i.e. that ${}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}$ is not a square matrix. Only square matrices can be orthogonal!

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Proof.

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Proof. Set ${}_{\mathcal{B}_V}[f]_{\mathcal{B}_U} = \left[\begin{array}{ccc} \mathbf{c}_1 & \dots & \mathbf{c}_m \end{array} \right]$.

1. the columns of the $n \times m$ matrix ${}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}$ form an orthonormal set of vectors in $\mathbb{R}^n$;
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Proof. Set ${}_{\mathcal{B}_V}[f]_{\mathcal{B}_U} = \begin{bmatrix} \mathbf{c}_1 & \dots & \mathbf{c}_m \end{bmatrix}$. We observe that

$$\begin{aligned} \left({}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}\right)^T {}_{\mathcal{B}_V}[f]_{\mathcal{B}_U} &= \begin{bmatrix} \mathbf{c}_1^T \\ \mathbf{c}_2^T \\ \vdots \\ \mathbf{c}_m^T \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_m \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{c}_1 \cdot \mathbf{c}_1 & \mathbf{c}_1 \cdot \mathbf{c}_2 & \dots & \mathbf{c}_1 \cdot \mathbf{c}_m \\ \mathbf{c}_2 \cdot \mathbf{c}_1 & \mathbf{c}_2 \cdot \mathbf{c}_2 & \dots & \mathbf{c}_2 \cdot \mathbf{c}_m \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{c}_m \cdot \mathbf{c}_1 & \mathbf{c}_m \cdot \mathbf{c}_2 & \dots & \mathbf{c}_m \cdot \mathbf{c}_m \end{bmatrix}. \end{aligned}$$

So, we see that (i) holds if and only if $\left({}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}\right)^T {}_{\mathcal{B}_V}[f]_{\mathcal{B}_U} = I_m$.

1. the columns of the $n \times m$ matrix ${}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}$ form an orthonormal set of vectors in $\mathbb{R}^n$;
2. for all $\mathbf{x}, \mathbf{y} \in U$, we have that $\langle f(\mathbf{x}), f(\mathbf{y}) \rangle_V = \langle \mathbf{x}, \mathbf{y} \rangle_U$.

Proof (continued). By Proposition 3.1, for all $\mathbf{x}, \mathbf{y} \in U$:

(1) $\langle \mathbf{x}, \mathbf{y} \rangle_U = [\mathbf{x}]_{\mathcal{B}_U} \cdot [\mathbf{y}]_{\mathcal{B}_U}$;

(2) $\langle f(\mathbf{x}), f(\mathbf{y}) \rangle_V = [f(\mathbf{x})]_{\mathcal{B}_V} \cdot [f(\mathbf{y})]_{\mathcal{B}_V}$.

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(2) $\langle f(\mathbf{x}), f(\mathbf{y}) \rangle_V = [f(\mathbf{x})]_{\mathcal{B}_V} \cdot [f(\mathbf{y})]_{\mathcal{B}_V}$.

Now, for all $\mathbf{x}, \mathbf{y} \in U$, we have that

$$\begin{aligned}
\langle f(\mathbf{x}), f(\mathbf{y}) \rangle_V &\overset{(2)}{=} [f(\mathbf{x})]_{\mathcal{B}_V} \cdot [f(\mathbf{y})]_{\mathcal{B}_V} \\
&= \left([f(\mathbf{x})]_{\mathcal{B}_V}\right)^T \, [f(\mathbf{y})]_{\mathcal{B}_V} \\
&= \left({}_{\mathcal{B}_V}[f]_{\mathcal{B}_U} \, [\mathbf{x}]_{\mathcal{B}_U}\right)^T \left({}_{\mathcal{B}_V}[f]_{\mathcal{B}_U} \, [\mathbf{y}]_{\mathcal{B}_U}\right) \\
&= [\mathbf{x}]_{\mathcal{B}_U}{}^T \left({}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}\right)^T \, {}_{\mathcal{B}_V}[f]_{\mathcal{B}_U} \, [\mathbf{y}]_{\mathcal{B}_U}.
\end{aligned}$$

1. the columns of the $n \times m$ matrix ${}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}$ form an orthonormal set of vectors in $\mathbb{R}^n$;
2. for all $\mathbf{x}, \mathbf{y} \in U$, we have that $\langle f(\mathbf{x}), f(\mathbf{y}) \rangle_V = \langle \mathbf{x}, \mathbf{y} \rangle_U$.

Proof (continued). Reminder: For all $\mathbf{x}, \mathbf{y} \in U$:

(1) $\langle \mathbf{x}, \mathbf{y} \rangle_U = [\mathbf{x}]_{\mathcal{B}_U} \cdot [\mathbf{y}]_{\mathcal{B}_U}$;

(2) $\langle f(\mathbf{x}), f(\mathbf{y}) \rangle_V = [f(\mathbf{x})]_{\mathcal{B}_V} \cdot [f(\mathbf{y})]_{\mathcal{B}_V}$.

$\langle f(\mathbf{x}), f(\mathbf{y}) \rangle_V = {[\mathbf{x}]_{\mathcal{B}_U}}^T \ ({}_{\mathcal{B}_V}[f]_{\mathcal{B}_U})^T \ {}_{\mathcal{B}_V}[f]_{\mathcal{B}_U} \ [\mathbf{y}]_{\mathcal{B}_U}$.

1. the columns of the $n \times m$ matrix ${}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}$ form an orthonormal set of vectors in $\mathbb{R}^n$;
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Proof (continued). Reminder: For all $\mathbf{x}, \mathbf{y} \in U$:

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(2) $\langle f(\mathbf{x}), f(\mathbf{y}) \rangle_V = [f(\mathbf{x})]_{\mathcal{B}_V} \cdot [f(\mathbf{y})]_{\mathcal{B}_V}$.

$\langle f(\mathbf{x}), f(\mathbf{y}) \rangle_V = [\mathbf{x}]_{\mathcal{B}_U}{}^T \left({}_{\mathcal{B}_V}[f]_{\mathcal{B}_U} \right)^T {}_{\mathcal{B}_V}[f]_{\mathcal{B}_U} \, [\mathbf{y}]_{\mathcal{B}_U}$.

Suppose first that (i) holds. Then $\left({}_{\mathcal{B}_V}[f]_{\mathcal{B}_U} \right)^T {}_{\mathcal{B}_V}[f]_{\mathcal{B}_U} = I_m$,

1. the columns of the $n \times m$ matrix ${}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}$ form an orthonormal set of vectors in $\mathbb{R}^n$;
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Proof (continued). Reminder: For all $\mathbf{x}, \mathbf{y} \in U$:

(1) $\langle \mathbf{x}, \mathbf{y} \rangle_U = [\mathbf{x}]_{\mathcal{B}_U} \cdot [\mathbf{y}]_{\mathcal{B}_U}$;

(2) $\langle f(\mathbf{x}), f(\mathbf{y}) \rangle_V = [f(\mathbf{x})]_{\mathcal{B}_V} \cdot [f(\mathbf{y})]_{\mathcal{B}_V}$.

$\langle f(\mathbf{x}), f(\mathbf{y}) \rangle_V = [\mathbf{x}]_{\mathcal{B}_U}{}^T \; ({}_{\mathcal{B}_V}[f]_{\mathcal{B}_U})^T \; {}_{\mathcal{B}_V}[f]_{\mathcal{B}_U} \; [\mathbf{y}]_{\mathcal{B}_U}$.

Suppose first that (i) holds. Then $({}_{\mathcal{B}_V}[f]_{\mathcal{B}_U})^T \; {}_{\mathcal{B}_V}[f]_{\mathcal{B}_U} = I_m$, and consequently, for all $\mathbf{x}, \mathbf{y} \in U$, we have that

$$\begin{aligned} \langle f(\mathbf{x}), f(\mathbf{y}) \rangle_V &= [\mathbf{x}]_{\mathcal{B}_U}{}^T \; \underbrace{({}_{\mathcal{B}_V}[f]_{\mathcal{B}_U})^T \; {}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}}_{=I_m} \; [\mathbf{y}]_{\mathcal{B}_U} \\ &= [\mathbf{x}]_{\mathcal{B}_U}{}^T \; [\mathbf{y}]_{\mathcal{B}_U} \\ &= [\mathbf{x}]_{\mathcal{B}_U} \cdot [\mathbf{y}]_{\mathcal{B}_U} \\ &\overset{(1)}{=} \langle \mathbf{x}, \mathbf{y} \rangle_U. \end{aligned}$$

Thus, (ii) holds.

1. the columns of the $n \times m$ matrix ${}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}$ form an orthonormal set of vectors in $\mathbb{R}^n$;
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Proof (continued). Reminder: ${}_{\mathcal{B}_V}[f]_{\mathcal{B}_U} = \begin{bmatrix} \mathbf{c}_1 & \dots & \mathbf{c}_m \end{bmatrix}$.

Suppose now that (ii) holds. Then for all $i, j \in \{1, \dots, m\}$, we have that

$$\begin{aligned}
\mathbf{e}_i^m \cdot \mathbf{e}_j^m &= [\mathbf{u}_i]_{\mathcal{B}_U} \cdot [\mathbf{u}_j]_{\mathcal{B}_U} \\
&\overset{(1)}{=} \langle \mathbf{u}_i, \mathbf{u}_j \rangle_U \\
&\overset{(ii)}{=} \langle f(\mathbf{u}_i), f(\mathbf{u}_j) \rangle_V \\
&\overset{(2)}{=} [f(\mathbf{u}_i)]_{\mathcal{B}_V} \cdot [f(\mathbf{u}_j)]_{\mathcal{B}_V} \\
&= \left({}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}\ [\mathbf{u}_i]_{\mathcal{B}_U}\right) \cdot \left({}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}\ [\mathbf{u}_j]_{\mathcal{B}_U}\right) \\
&= \left({}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}\ \mathbf{e}_i^m\right) \cdot \left({}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}\ \mathbf{e}_j^m\right) \\
&= \mathbf{c}_i \cdot \mathbf{c}_j.
\end{aligned}$$

1. the columns of the $n \times m$ matrix ${}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}$ form an orthonormal set of vectors in $\mathbb{R}^n$;
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Proof (continued). Reminder: ${}_{\mathcal{B}_V}[f]_{\mathcal{B}_U} = \begin{bmatrix} \mathbf{c}_1 & \dots & \mathbf{c}_m \end{bmatrix}$; for all $i, j \in \{1, \dots, m\}$, $\mathbf{e}_i^m \cdot \mathbf{e}_j^m = \mathbf{c}_i \cdot \mathbf{c}_j$.

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Proof (continued). Reminder: ${}_{\mathcal{B}_V}[f]_{\mathcal{B}_U} = \begin{bmatrix} \mathbf{c}_1 & \dots & \mathbf{c}_m \end{bmatrix}$; for all $i, j \in \{1, \dots, m\}$, $\mathbf{e}_i^m \cdot \mathbf{e}_j^m = \mathbf{c}_i \cdot \mathbf{c}_j$.

This implies that $\{\mathbf{c}_1, \dots, \mathbf{c}_m\}$ form an orthonormal set of vectors in $\mathbb{R}^n$, i.e. (i) holds.

Theorem 3.2

Let U and V be non-trivial, finite-dimensional vector spaces over $\mathbb{R}$. Assume that U is equipped with a scalar product $\langle \cdot, \cdot \rangle_U$ and the induced norm $|| \cdot ||_U$, and that V is equipped with a scalar product $\langle \cdot, \cdot \rangle_V$ and the induced norm $|| \cdot ||_V$. Let $\mathcal{B}_U = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ and $\mathcal{B}_V = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be orthonormal bases for U and V, respectively, and let $f : U \to V$ be a linear transformation. Then the following two statements are equivalent:

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[a]However, despite Theorem 2.1, this does not necessarily mean that the matrix ${}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}$ is orthogonal. This is because ${}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}$ is an $n \times m$ matrix, and it is possible that $m \neq n$, i.e. that ${}_{\mathcal{B}_V}[f]_{\mathcal{B}_U}$ is not a square matrix. Only square matrices can be orthogonal!