Linear Algebra 2

Lecture #12

Gram-Schmidt orthogonalization. Orthogonal complements

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February 23, 2023

Definition

For a vector space V over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$, we say that vectors $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal*, and we write $\mathbf{x} \perp \mathbf{y}$, provided that $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

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- As we saw in Lecture 11:
  - the orthogonality relation is symmetric, i.e. $\mathbf{x} \perp \mathbf{y}$ iff $\mathbf{y} \perp \mathbf{x}$;
  - $\mathbf{0}$ is orthogonal to every vector in V.

Definition

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$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) \quad := \quad \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}.$$

- Note: $\mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ is a scalar multiple of $\mathbf{u}$.

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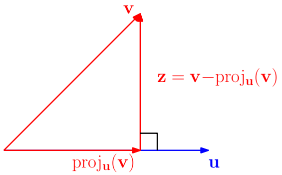
Proposition 1.1

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle\cdot,\cdot\rangle$. Let $\mathbf{u}$ be a non-zero vector in V, let $\mathbf{v}$ be any vector in V, and set $\mathbf{z} := \mathbf{v} - \mathrm{proj}_{\mathbf{u}}(\mathbf{v})$. Then $\mathbf{z} \perp \mathbf{u}$.



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Proof. We have that

$$\begin{aligned}
\langle \mathbf{z}, \mathbf{u} \rangle &= \langle \mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v}), \mathbf{u} \rangle \\
&= \langle \mathbf{v} - \tfrac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}, \mathbf{u} \rangle \\
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&= \langle \mathbf{v}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{u} \rangle \\
&= 0,
\end{aligned}$$

and consequently, $\mathbf{z} \perp \mathbf{u}$. Q.E.D.

Definition

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the norm $|| \cdot ||$ induced by $\langle \cdot, \cdot \rangle$.

- An *orthogonal set of vectors* in V is a collection of pairwise orthogonal vectors in V.
- An *orthonormal set of vectors* is an orthogonal set of vectors that satisfies the additional property that all vectors in this set are of length 1 (i.e. every vector $\mathbf{v}$ in the set satisfies $||\mathbf{v}|| = 1$).
- An *orthogonal basis* (resp. *orthonormal basis*) for V is an orthogonal (resp. orthonormal) set in V that is also a basis of V.

Theorem 2.1

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an orthogonal basis of V. Then for all $\mathbf{v} \in V$, we have that

$$\mathbf{v} \;=\; \sum_{i=1}^{n} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i \;=\; \sum_{i=1}^{n} \mathrm{proj}_{\mathbf{u}_i}(\mathbf{v}).$$

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Corollary 2.2

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the norm $|| \cdot ||$ induced by $\langle \cdot, \cdot \rangle$. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be an orthonormal basis of V. Then for all $\mathbf{v} \in V$, we have that

$$\mathbf{v} \;=\; \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{u}_i \rangle \, \mathbf{u}_i$$

- Coefficients $\langle \mathbf{v}, \mathbf{u}_i \rangle$ from Corollary 2.2 are called the *Fourier coefficients*.

Proof of Theorem 2.1.

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$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i.$$

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$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i.$$

Now, fix any $j \in \{1, \dots, n\}$. We then have that

$$\begin{aligned} \langle \mathbf{v}, \mathbf{u}_j \rangle &= \langle \sum_{i=1}^{n} \alpha_i \mathbf{u}_i, \mathbf{u}_j \rangle \\ &= \sum_{i=1}^{n} \alpha_i \langle \mathbf{u}_i, \mathbf{u}_j \rangle \\ &\overset{(*)}{=} \alpha_j \langle \mathbf{u}_j, \mathbf{u}_j \rangle, \end{aligned}$$

where (*) follows from the fact that $\mathbf{u}_1, \dots, \mathbf{u}_n$ are pairwise orthogonal.

Proof of Theorem 2.1 (continued). Reminder: $\langle \mathbf{v}, \mathbf{u}_j \rangle = \alpha_j \langle \mathbf{u}_j, \mathbf{u}_j \rangle$.

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Since $\mathbf{u}_j \neq \mathbf{0}$ (because $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is a basis of V), we see that $\langle \mathbf{u}_j, \mathbf{u}_j \rangle \neq 0$, and we deduce that

$$\alpha_j = \frac{\langle \mathbf{v}, \mathbf{u}_j \rangle}{\langle \mathbf{u}_j, \mathbf{u}_j \rangle}.$$

Proof of Theorem 2.1 (continued). Reminder: $\langle \mathbf{v}, \mathbf{u}_j \rangle = \alpha_j \langle \mathbf{u}_j, \mathbf{u}_j \rangle$.

Since $\mathbf{u}_j \neq \mathbf{0}$ (because $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis of V), we see that $\langle \mathbf{u}_j, \mathbf{u}_j \rangle \neq 0$, and we deduce that

$$\alpha_j = \frac{\langle \mathbf{v}, \mathbf{u}_j \rangle}{\langle \mathbf{u}_j, \mathbf{u}_j \rangle}.$$

Since $j \in \{1, \ldots, n\}$ was chosen arbitrarily, we now deduce that

$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i = \sum_{i=1}^{n} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i.$$

Q.E.D.

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Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an orthogonal basis of V. Then for all $\mathbf{v} \in V$, we have that

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Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle\cdot,\cdot\rangle$ and the norm $||\cdot||$ induced by $\langle\cdot,\cdot\rangle$. Let $\{\mathbf{u}_1,\ldots,\mathbf{u}_n\}$ be an orthonormal basis of V. Then for all $\mathbf{v}\in V$, we have that

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Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the norm $|| \cdot ||$ induced by $\langle \cdot, \cdot \rangle$. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be an orthonormal basis of V. Then for all $\mathbf{v} \in V$, we have that

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Proof.

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$$\mathbf{v} = \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{u}_i \rangle \, \mathbf{u}_i$$

Proof. Since $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an orthonormal basis of V, we know that $||\mathbf{u}_1|| = \dots = ||\mathbf{u}_n|| = 1$, and consequently (by the construction of $|| \cdot ||$), we have that $\langle \mathbf{u}_1, \mathbf{u}_1 \rangle = \dots = \langle \mathbf{u}_n, \mathbf{u}_n \rangle = 1$. The result now follows immediately from Theorem 2.1. Q.E.D.

- Our next goal is to describe the "Gram-Schmidt orthogonalization process," which gives a recipe for transforming an arbitrary basis of a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product and the norm induced by it, into an orthogonal (and even orthonormal) basis.

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- But first, we need a couple of technical propositions.

Proposition 2.3

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthogonal set of non-zero vectors in V. Then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is linearly independent.

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Proof. Fix scalars $\alpha_1, \dots, \alpha_k$ such that

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Now, note that

$$\begin{aligned} \langle \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k, \mathbf{u}_i \rangle &= \alpha_1 \langle \mathbf{u}_1, \mathbf{u}_i \rangle + \dots + \alpha_k \langle \mathbf{u}_k, \mathbf{u}_i \rangle \\ &\overset{(*)}{=} \alpha_i \langle \mathbf{u}_i, \mathbf{u}_i \rangle, \end{aligned}$$

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Now, note that

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where (*) follows from the fact that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal set. So, $\alpha_i \langle \mathbf{u}_i, \mathbf{u}_i \rangle = 0$. Since $\mathbf{u}_i \neq \mathbf{0}$, we see that $\langle \mathbf{u}_i, \mathbf{u}_i \rangle \neq 0$; consequently, $\alpha_i = 0$. Q.E.D.

Proposition 2.4

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle\cdot,\cdot\rangle$. Let $\{\mathbf{u}_1,\dots,\mathbf{u}_k\}$ be an orthogonal set of non-zero vectors in V. Let $\mathbf{v}\in V$, and set $\mathbf{y} := \sum_{i=1}^{k} \text{proj}_{\mathbf{u}_i}(\mathbf{v}) = \sum_{i=1}^{k} \frac{\langle\mathbf{v},\mathbf{u}_i\rangle}{\langle\mathbf{u}_i,\mathbf{u}_i\rangle}\mathbf{u}_i$ and $\mathbf{z} := \mathbf{v}-\mathbf{y}$.

(a) $\{\mathbf{u}_1,\dots,\mathbf{u}_k,\mathbf{z}\}$ is an orthogonal set of vectors;

(b) $\mathbf{z}=\mathbf{0}$ if and only if $\mathbf{v}\in\text{Span}(\mathbf{u}_1,\dots,\mathbf{u}_k)$;

(c) $\text{Span}(\mathbf{u}_1,\dots,\mathbf{u}_k,\mathbf{v}) = \text{Span}(\mathbf{u}_1,\dots,\mathbf{u}_k,\mathbf{z})$.

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(a) $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z}\}$ is an orthogonal set of vectors;

(b) $\mathbf{z} = \mathbf{0}$ if and only if $\mathbf{v} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$;

(c) $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}) = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z})$.

Proof. First of all, we remark that Proposition 2.3 guarantees that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a linearly independent set, and we deduce that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$.

Proof of Proposition 2.4 (continued).
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Since $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z}\}$ is an orthogonal set of vectors, it suffices to show that $\langle \mathbf{z}, \mathbf{u}_j \rangle = 0$ for all $j \in \{1, \dots, k\}$.

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$$\begin{aligned}
\langle \mathbf{z}, \mathbf{u}_j \rangle &= \Big\langle \mathbf{v} - \sum_{i=1}^{k} \tfrac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i, \mathbf{u}_j \Big\rangle \\
&= \langle \mathbf{v}, \mathbf{u}_j \rangle - \sum_{i=1}^{k} \tfrac{\langle \mathbf{v}, \mathbf{u}_j \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \langle \mathbf{u}_i, \mathbf{u}_j \rangle \\
&\overset{(*)}{=} \langle \mathbf{v}, \mathbf{u}_j \rangle - \tfrac{\langle \mathbf{v}, \mathbf{u}_j \rangle}{\langle \mathbf{u}_j, \mathbf{u}_j \rangle} \langle \mathbf{u}_j, \mathbf{u}_j \rangle \\
&= \langle \mathbf{v}, \mathbf{u}_j \rangle - \langle \mathbf{v}, \mathbf{u}_j \rangle \\
&= 0.
\end{aligned}$$

where (*) follows from the fact that $\{\mathbf{u_1}, \dots, \mathbf{u}_k\}$ is an orthogonal set. This proves (a).

Proof of Proposition 2.4 (continued).
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So, it is enough to show that $\mathbf{v} \in \mathrm{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ iff $\mathbf{v} = \sum_{i=1}^{k} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i$.

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If $\mathbf{v} = \sum_{i=1}^{k} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i$, then $\mathbf{v}$ is a linear combination of the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$, and consequently, $\mathbf{v} \in \mathrm{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$.

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So, it is enough to show that $\mathbf{v} \in \text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ iff $\mathbf{v} = \sum_{i=1}^{k} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i$.

If $\mathbf{v} = \sum_{i=1}^{k} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i$, then $\mathbf{v}$ is a linear combination of the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$, and consequently, $\mathbf{v} \in \text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$.

On the other hand, if $\mathbf{v} \in \text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$, then Theorem 2.1 guarantees $\mathbf{v} = \sum_{i=1}^{k} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i$. This proves (b).

Proof of Proposition 2.4 (continued).
(c) WTS $\mathrm{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}) = \mathrm{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{z})$.

Proof of Proposition 2.4 (continued).
(c) WTS $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}) = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z})$.

Fix any vector $\mathbf{x} \in V$. WTS $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v})$ iff $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z})$.

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Suppose first that $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v})$. Then there exist scalars $\alpha_1, \dots, \alpha_k, \beta$ s.t. $\mathbf{x} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k + \beta \mathbf{v}$.

Proof of Proposition 2.4 (continued).
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$$\begin{aligned}
\mathbf{x} &= \alpha_1\mathbf{u}_1 + \cdots + \alpha_k\mathbf{u}_k + \beta\mathbf{v} \\
&= \Big(\sum_{i=1}^{k} \alpha_i\mathbf{u}_i\Big) + \beta(\mathbf{y} + \mathbf{z}) \\
&= \Big(\sum_{i=1}^{k} \alpha_i\mathbf{u}_i\Big) + \beta\Big(\Big(\sum_{i=1}^{k} \tfrac{\langle \mathbf{v}, \mathbf{u}_i\rangle}{\langle \mathbf{u}_i, \mathbf{u}_i\rangle}\mathbf{u}_i\Big) + \mathbf{z}\Big) \\
&= \Big(\sum_{i=1}^{k}\Big(\alpha_i + \beta\tfrac{\langle \mathbf{v}, \mathbf{u}_i\rangle}{\langle \mathbf{u}_i, \mathbf{u}_i\rangle}\Big)\mathbf{u}_i\Big) + \beta\mathbf{z},
\end{aligned}$$

and we deduce that $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{z})$.

Proof of Proposition 2.4 (continued).
(c) WTS $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}) = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z})$.

Fix any vector $\mathbf{x} \in V$. WTS $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v})$ iff $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z})$.

Suppose first that $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v})$. Then there exist scalars $\alpha_1, \dots, \alpha_k, \beta$ s.t. $\mathbf{x} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k + \beta \mathbf{v}$. But now

$$\begin{aligned}
\mathbf{x} &= \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k + \beta \mathbf{v} \\
&= \Big(\sum_{i=1}^{k} \alpha_i \mathbf{u}_i\Big) + \beta(\mathbf{y} + \mathbf{z}) \\
&= \Big(\sum_{i=1}^{k} \alpha_i \mathbf{u}_i\Big) + \beta\bigg(\Big(\sum_{i=1}^{k} \tfrac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i\Big) + \mathbf{z}\bigg) \\
&= \bigg(\sum_{i=1}^{k} \Big(\alpha_i + \beta \tfrac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle}\Big) \mathbf{u}_i\bigg) + \beta \mathbf{z},
\end{aligned}$$

and we deduce that $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z})$.

The other direction is similar. This proves (c).

Proposition 2.3

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthogonal set of non-zero vectors in V. Then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is linearly independent.

Proposition 2.4

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthogonal set of non-zero vectors in V. Let $\mathbf{v} \in V$, and set $\mathbf{y} := \sum_{i=1}^{k} \mathrm{proj}_{\mathbf{u}_i}(\mathbf{v}) = \sum_{i=1}^{k} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i$ and $\mathbf{z} := \mathbf{v} - \mathbf{y}$.

a. $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{z}\}$ is an orthogonal set of vectors;
b. $\mathbf{z} = \mathbf{0}$ if and only if $\mathbf{v} \in \mathrm{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$;
c. $\mathrm{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}) = \mathrm{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{z})$.

The Gram-Schmidt orthogonalization process (version 1)

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the norm $|| \cdot ||$ induced by $\langle \cdot, \cdot \rangle$. Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be a linearly independent set of vectors in V. $\forall \ell \in \{1, \dots, k\}$, set

$$\mathbf{u}_\ell \;=\; \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \text{proj}_{\mathbf{u}_i}(\mathbf{v}_\ell) \;=\; \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \frac{\langle \mathbf{v}_\ell, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i.$$

Then $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$. Moreover, $\{\frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||}\}$ is an orthonormal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

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$$\mathbf{u}_\ell \quad = \quad \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \text{proj}_{\mathbf{u}_i}(\mathbf{v}_\ell) \quad = \quad \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \frac{\langle \mathbf{v}_\ell, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i.$$

Then $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$. Moreover, $\{\frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||}\}$ is an orthonormal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

- $\mathbf{u}_1 = \mathbf{v}_1$;
- $\mathbf{u}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{u}_1}(\mathbf{v}_2)$;
- $\mathbf{u}_3 = \mathbf{v}_3 - \Big(\text{proj}_{\mathbf{u}_1}(\mathbf{v}_3) + \text{proj}_{\mathbf{u}_2}(\mathbf{v}_3)\Big)$;

  $\vdots$
- $\mathbf{u}_k = \mathbf{v}_k - \Big(\text{proj}_{\mathbf{u}_1}(\mathbf{v}_k) + \text{proj}_{\mathbf{u}_2}(\mathbf{v}_k) + \cdots + \text{proj}_{\mathbf{u}_{k-1}}(\mathbf{v}_k)\Big)$.

- First an example, then the proof.

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Example 2.6

Consider the vectors following vectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix}.$$

in $\mathbb{R}^4$, and set $U := \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Using the Gram-Schmidt orthogonalization process (version 1):

(a) find an orthogonal basis of U (with respect to the standard scalar product $\cdot$ in $\mathbb{R}^4$).

(b) find an orthonormal basis of U (with respect to the standard scalar product $\cdot$ in $\mathbb{R}^4$ and the norm $||\cdot||$ induced by it).

Solution of Example 2.6

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- $\mathbf{u}_1 := \mathbf{v}_1 = \begin{bmatrix} 3 & 1 & -1 & 3 \end{bmatrix}^T$

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- $\mathbf{u}_2 := \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} 1 & 3 & 3 & -1 \end{bmatrix}^T$

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- $\mathbf{u}_3 := \mathbf{v}_3 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 \right) = \begin{bmatrix} -3 & 1 & 1 & 3 \end{bmatrix}^T.$

Solution of Example 2.6

- $\mathbf{u}_1 := \mathbf{v}_1 = \begin{bmatrix} 3 & 1 & -1 & 3 \end{bmatrix}^T$
- $\mathbf{u}_2 := \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} 1 & 3 & 3 & -1 \end{bmatrix}^T$
- $\mathbf{u}_3 := \mathbf{v}_3 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2\right) = \begin{bmatrix} -3 & 1 & 1 & 3 \end{bmatrix}^T.$

So, $\mathcal{B} := \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$ is an orthogonal basis of U.

To obtain an orthonormal basis of U, we "normalize" the vectors from the orthogonal basis $\mathcal{B}$ of U from part (a), that is, we rescale them so that their length becomes 1.

Solution of Example 2.6

- $\mathbf{u}_1 := \mathbf{v}_1 = \begin{bmatrix} 3 & 1 & -1 & 3 \end{bmatrix}^T$
- $\mathbf{u}_2 := \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 = \begin{bmatrix} 1 & 3 & 3 & -1 \end{bmatrix}^T$
- $\mathbf{u}_3 := \mathbf{v}_3 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 \right) = \begin{bmatrix} -3 & 1 & 1 & 3 \end{bmatrix}^T.$

So, $\mathcal{B} := \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$ is an orthogonal basis of U.

To obtain an orthonormal basis of U, we “normalize” the vectors from the orthogonal basis $\mathcal{B}$ of U from part (a), that is, we rescale them so that their length becomes 1.

We compute $||\mathbf{u}_1|| = 2\sqrt{5}$, $||\mathbf{u}_2|| = 2\sqrt{5}$, $||\mathbf{u}_3|| = 2\sqrt{5}$.

Solution of Example 2.6 (continued).

We now see that

$$\begin{aligned}
\mathcal{C} \quad &:= \quad \left\{\frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \frac{\mathbf{u}_2}{||\mathbf{u}_2||}, \frac{\mathbf{u}_3}{||\mathbf{u}_3||}\right\} \\
&= \quad \left\{\frac{1}{2\sqrt{5}}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \frac{1}{2\sqrt{5}}\begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \frac{1}{2\sqrt{5}}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}\right\}
\end{aligned}$$

is an orthonormal basis of U.

The Gram-Schmidt orthogonalization process (version 1)

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the norm $|| \cdot ||$ induced by $\langle \cdot, \cdot \rangle$. Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be a linearly independent set of vectors in V. $\forall \ell \in \{1, \dots, k\}$, set

$$\mathbf{u}_\ell \;=\; \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \text{proj}_{\mathbf{u}_i}(\mathbf{v}_\ell) \;=\; \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \frac{\langle \mathbf{v}_\ell, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i.$$

Then $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$. Moreover, $\{\frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||}\}$ is an orthonormal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

- $\mathbf{u}_1 = \mathbf{v}_1$;
- $\mathbf{u}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{u}_1}(\mathbf{v}_2)$;
- $\mathbf{u}_3 = \mathbf{v}_3 - \Big(\text{proj}_{\mathbf{u}_1}(\mathbf{v}_3) + \text{proj}_{\mathbf{u}_2}(\mathbf{v}_3)\Big)$;

  $\vdots$
- $\mathbf{u}_k = \mathbf{v}_k - \Big(\text{proj}_{\mathbf{u}_1}(\mathbf{v}_k) + \text{proj}_{\mathbf{u}_2}(\mathbf{v}_k) + \dots + \text{proj}_{\mathbf{u}_{k-1}}(\mathbf{v}_k)\Big)$.

Proof (outline).

Proof (outline). If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is orthogonal basis of $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$, then obviously, $\{\frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \ldots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||}\}$ is an orthonormal basis of $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ (since we simply rescaled each vector so that its length is 1).

Proof (outline). If $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is orthogonal basis of Span$(\mathbf{v}_1, \dots, \mathbf{v}_k)$, then obviously, $\{\frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||}\}$ is an orthonormal basis of Span$(\mathbf{v}_1, \dots, \mathbf{v}_k)$ (since we simply rescaled each vector so that its length is 1).

Let us prove that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of Span$(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

Proof (outline). If $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is orthogonal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$, then obviously, $\{\frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||}\}$ is an orthonormal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ (since we simply rescaled each vector so that its length is 1).

Let us prove that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$. For each $\ell \in \{1, \dots, k\}$, we let $U_\ell := \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_\ell)$, and we prove that $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell\}$ is an orthogonal basis of U_ℓ.

Proof (outline). If $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is orthogonal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$, then obviously, $\{\frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||}\}$ is an orthonormal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ (since we simply rescaled each vector so that its length is 1).

Let us prove that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$. For each $\ell \in \{1, \dots, k\}$, we let $U_\ell := \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_\ell)$, and we prove that $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell\}$ is an orthogonal basis of U_ℓ. Obviously, this is enough, because for $k = \ell$, we get that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of $U_k = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

Proof (outline). If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is orthogonal basis of $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$, then obviously, $\{\frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \ldots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||}\}$ is an orthonormal basis of $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ (since we simply rescaled each vector so that its length is 1).

Let us prove that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$. For each $\ell \in \{1, \ldots, k\}$, we let $U_\ell := \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_\ell)$, and we prove that $\{\mathbf{u}_1, \ldots, \mathbf{u}_\ell\}$ is an orthogonal basis of U_ℓ. Obviously, this is enough, because for $k = \ell$, we get that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of $U_k = \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$.

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, we see that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are all non-zero, and in particular, $\{\mathbf{v}_1\}$ is linearly independent.

Proof (outline). If $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is orthogonal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$, then obviously, $\{\frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||}\}$ is an orthonormal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ (since we simply rescaled each vector so that its length is 1).

Let us prove that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$. For each $\ell \in \{1, \dots, k\}$, we let $U_\ell := \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_\ell)$, and we prove that $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell\}$ is an orthogonal basis of U_ℓ. Obviously, this is enough, because for $k = \ell$, we get that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of $U_k = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

Since $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent, we see that $\mathbf{v}_1, \dots, \mathbf{v}_k$ are all non-zero, and in particular, $\{\mathbf{v}_1\}$ is linearly independent. Since $U_1 = \text{Span}(\mathbf{v}_1)$ and $\mathbf{u}_1 = \mathbf{v}_1$, we deduce that $\{\mathbf{u}_1\}$ is a basis of U_1.

Proof (outline, continued).

Reminder: $U_\ell := \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_\ell)$ $\forall \ell \in \{1, \dots, k\}$.

Proof (outline, continued).
Reminder: $U_\ell := \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_\ell)\ \forall \ell \in \{1, \ldots, k\}$.

Now, fix $\ell \in \{1, \ldots, k-1\}$, and assume inductively that $\{\mathbf{u}_1, \ldots, \mathbf{u}_\ell\}$ is an orthogonal basis of U_ℓ. WTS $\{\mathbf{u}_1, \ldots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}\}$ is an orthogonal basis of $U_{\ell+1}$.

Proof (outline, continued).
Reminder: $U_\ell := \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_\ell)$ $\forall \ell \in \{1, \ldots, k\}$.

Now, fix $\ell \in \{1, \ldots, k-1\}$, and assume inductively that $\{\mathbf{u}_1, \ldots, \mathbf{u}_\ell\}$ is an orthogonal basis of U_ℓ. WTS $\{\mathbf{u}_1, \ldots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}\}$ is an orthogonal basis of $U_{\ell+1}$. Since $\{\mathbf{u}_1, \ldots, \mathbf{u}_\ell\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\}$ are two bases of U_ℓ, it is clear that
$\text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_\ell, \mathbf{v}_{\ell+1}) = \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_\ell, \mathbf{v}_{\ell+1}) = U_{\ell+1}$.

Proof (outline, continued).
Reminder: $U_\ell := \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_\ell)\ \forall \ell \in \{1, \dots, k\}$.

Now, fix $\ell \in \{1, \dots, k-1\}$, and assume inductively that $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell\}$ is an orthogonal basis of U_ℓ. WTS $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}\}$ is an orthogonal basis of $U_{\ell+1}$. Since $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell\}$ and $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ are two bases of U_ℓ, it is clear that $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{v}_{\ell+1}) = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_\ell, \mathbf{v}_{\ell+1}) = U_{\ell+1}$. On the other hand, by the construction of $\mathbf{u}_{\ell+1}$ and by Proposition 2.4(c), we have that $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{v}_{\ell+1}) = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1})$.

Proof (outline, continued).
Reminder: $U_\ell := \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_\ell)$ $\forall \ell \in \{1, \dots, k\}$.

Now, fix $\ell \in \{1, \dots, k-1\}$, and assume inductively that $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell\}$ is an orthogonal basis of U_ℓ. WTS $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}\}$ is an orthogonal basis of $U_{\ell+1}$. Since $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell\}$ and $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ are two bases of U_ℓ, it is clear that $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{v}_{\ell+1}) = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_\ell, \mathbf{v}_{\ell+1}) = U_{\ell+1}$. On the other hand, by the construction of $\mathbf{u}_{\ell+1}$ and by Proposition 2.4(c), we have that $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{v}_{\ell+1}) = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1})$. So, $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}) = U_{\ell+1}$.

Proof (outline, continued).
Reminder: $U_\ell := \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_\ell)\ \forall \ell \in \{1, \dots, k\}$.

Now, fix $\ell \in \{1, \dots, k-1\}$, and assume inductively that $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell\}$ is an orthogonal basis of U_ℓ. WTS $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}\}$ is an orthogonal basis of $U_{\ell+1}$. Since $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell\}$ and $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ are two bases of U_ℓ, it is clear that $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{v}_{\ell+1}) = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_\ell, \mathbf{v}_{\ell+1}) = U_{\ell+1}$. On the other hand, by the construction of $\mathbf{u}_{\ell+1}$ and by Proposition 2.4(c), we have that $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{v}_{\ell+1}) = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1})$. So, $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}) = U_{\ell+1}$. Since $\dim(U_{\ell+1}) = \ell + 1$, the fact that $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}\}$ spans $U_{\ell+1}$ implies that $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}\}$ is in fact a basis of $U_{\ell+1}$.

Proof (outline, continued).
Reminder: $U_\ell := \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_\ell)$ $\forall \ell \in \{1, \dots, k\}$.

Now, fix $\ell \in \{1, \dots, k-1\}$, and assume inductively that $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell\}$ is an orthogonal basis of U_ℓ. WTS $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}\}$ is an orthogonal basis of $U_{\ell+1}$. Since $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell\}$ and $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ are two bases of U_ℓ, it is clear that $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{v}_{\ell+1}) = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_\ell, \mathbf{v}_{\ell+1}) = U_{\ell+1}$. On the other hand, by the construction of $\mathbf{u}_{\ell+1}$ and by Proposition 2.4(c), we have that $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{v}_{\ell+1}) = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1})$. So, $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}) = U_{\ell+1}$. Since $\dim(U_{\ell+1}) = \ell + 1$, the fact that $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}\}$ spans $U_{\ell+1}$ implies that $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}\}$ is in fact a basis of $U_{\ell+1}$. By the induction hypothesis, vectors $\mathbf{u}_1, \dots, \mathbf{u}_\ell$ are pairwise orthogonal non-zero vectors, and so by the costruction of $\mathbf{u}_{\ell+1}$ and by Proposition 2.4(a), we have that $\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}$ are pairwise orthogonal.

Proof (outline, continued).
Reminder: $U_\ell := \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_\ell)$ $\forall \ell \in \{1, \dots, k\}$.

Now, fix $\ell \in \{1, \dots, k-1\}$, and assume inductively that $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell\}$ is an orthogonal basis of U_ℓ. WTS $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}\}$ is an orthogonal basis of $U_{\ell+1}$. Since $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell\}$ and $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ are two bases of U_ℓ, it is clear that $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{v}_{\ell+1}) = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_\ell, \mathbf{v}_{\ell+1}) = U_{\ell+1}$. On the other hand, by the construction of $\mathbf{u}_{\ell+1}$ and by Proposition 2.4(c), we have that $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{v}_{\ell+1}) = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1})$. So, $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}) = U_{\ell+1}$. Since $\dim(U_{\ell+1}) = \ell + 1$, the fact that $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}\}$ spans $U_{\ell+1}$ implies that $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}\}$ is in fact a basis of $U_{\ell+1}$. By the induction hypothesis, vectors $\mathbf{u}_1, \dots, \mathbf{u}_\ell$ are pairwise orthogonal non-zero vectors, and so by the costruction of $\mathbf{u}_{\ell+1}$ and by Proposition 2.4(a), we have that $\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}$ are pairwise orthogonal. So, $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell, \mathbf{u}_{\ell+1}\}$ is an orthogonal basis of $U_{\ell+1}$. This completes the induction. Q.E.D.

The Gram-Schmidt orthogonalization process (version 1)

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the norm $|| \cdot ||$ induced by $\langle \cdot, \cdot \rangle$. Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be a linearly independent set of vectors in V. $\forall \ell \in \{1, \dots, k\}$, set

$$\mathbf{u}_\ell \quad = \quad \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \text{proj}_{\mathbf{u}_i}(\mathbf{v}_\ell) \quad = \quad \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \frac{\langle \mathbf{v}_\ell, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i.$$

Then $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$. Moreover, $\{\frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||}\}$ is an orthonormal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

- $\mathbf{u}_1 = \mathbf{v}_1$;
- $\mathbf{u}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{u}_1}(\mathbf{v}_2)$;
- $\mathbf{u}_3 = \mathbf{v}_3 - \Big(\text{proj}_{\mathbf{u}_1}(\mathbf{v}_3) + \text{proj}_{\mathbf{u}_2}(\mathbf{v}_3)\Big)$;

  $\vdots$
- $\mathbf{u}_k = \mathbf{v}_k - \Big(\text{proj}_{\mathbf{u}_1}(\mathbf{v}_k) + \text{proj}_{\mathbf{u}_2}(\mathbf{v}_k) + \dots + \text{proj}_{\mathbf{u}_{k-1}}(\mathbf{v}_k)\Big)$.

Corollary 2.5

Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the norm $|| \cdot ||$ induced by $\langle \cdot, \cdot \rangle$. Let U be a subspace of V. Then all the following hold:

(a) U has an orthogonal basis;

(b) any orthogonal basis of U can be extended to an orthogonal basis of V;

(c) U has an orthonormal basis;

(d) any orthonormal basis of U can be extended to an orthonormal basis of V.

Proof.

Corollary 2.5

Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the norm $|| \cdot ||$ induced by $\langle \cdot, \cdot \rangle$. Let U be a subspace of V. Then all the following hold:

a. U has an orthogonal basis;
b. any orthogonal basis of U can be extended to an orthogonal basis of V;
c. U has an orthonormal basis;
d. any orthonormal basis of U can be extended to an orthonormal basis of V.

Proof. Consider any basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of U.

Corollary 2.5

Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the norm $|| \cdot ||$ induced by $\langle \cdot, \cdot \rangle$. Let U be a subspace of V. Then all the following hold:

- (a) U has an orthogonal basis;
- (b) any orthogonal basis of U can be extended to an orthogonal basis of V;
- (c) U has an orthonormal basis;
- (d) any orthonormal basis of U can be extended to an orthonormal basis of V.

Proof. Consider any basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of U. Then the Gram-Schmidt orthogonalization process applied to the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ yields a sequence of vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ such that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal and $\{\frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||}\}$ an orthonormal basis of $U = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$. This proves (a) and (c).

1. U has an orthogonal basis;
2. any orthogonal basis of U can be extended to an orthogonal basis of V;
3. U has an orthonormal basis;
4. any orthonormal basis of U can be extended to an orthonormal basis of V.

Proof (continued).

(a) U has an orthogonal basis;
(b) any orthogonal basis of U can be extended to an orthogonal basis of V;
(c) U has an orthonormal basis;
(d) any orthonormal basis of U can be extended to an orthonormal basis of V.

Proof (continued). For (b), consider any orthogonal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of U, and extend it to a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ of V.

(a) U has an orthogonal basis;
(b) any orthogonal basis of U can be extended to an orthogonal basis of V;
(c) U has an orthonormal basis;
(d) any orthonormal basis of U can be extended to an orthonormal basis of V.

Proof (continued). For (b), consider any orthogonal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of U, and extend it to a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ of V. We then apply the Gram-Schmidt orthogonaliation process to the sequence $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n$, and we obtain some sequence $\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n$ such that $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of V.

(a) U has an orthogonal basis;

(b) any orthogonal basis of U can be extended to an orthogonal basis of V;

(c) U has an orthonormal basis;

(d) any orthonormal basis of U can be extended to an orthonormal basis of V.

Proof (continued). For (b), consider any orthogonal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of U, and extend it to a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ of V. We then apply the Gram-Schmidt orthogonaliation process to the sequence $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n$, and we obtain some sequence $\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n$ such that $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of V. However, since $\mathbf{v}_1, \dots, \mathbf{v}_k$ were pairwise orthogonal to begin with, we see from the description of the Gram-Schmidt orthogonalization process that $\mathbf{u}_1 = \mathbf{v}_1, \dots, \mathbf{u}_k = \mathbf{v}_k$.

(a) U has an orthogonal basis;

(b) any orthogonal basis of U can be extended to an orthogonal basis of V;

(c) U has an orthonormal basis;

(d) any orthonormal basis of U can be extended to an orthonormal basis of V.

Proof (continued). For (b), consider any orthogonal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of U, and extend it to a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ of V. We then apply the Gram-Schmidt orthogonaliation process to the sequence $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n$, and we obtain some sequence $\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n$ such that $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of V. However, since $\mathbf{v}_1, \dots, \mathbf{v}_k$ were pairwise orthogonal to begin with, we see from the description of the Gram-Schmidt orthogonalization process that $\mathbf{u}_1 = \mathbf{v}_1, \dots, \mathbf{u}_k = \mathbf{v}_k$. So, the orthogonal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ of V extends the orthogonal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of U. This proves (b).

(a) U has an orthogonal basis;
(b) any orthogonal basis of U can be extended to an orthogonal basis of V;
(c) U has an orthonormal basis;
(d) any orthonormal basis of U can be extended to an orthonormal basis of V.

Proof (continued).

(a) U has an orthogonal basis;

(b) any orthogonal basis of U can be extended to an orthogonal basis of V;

(c) U has an orthonormal basis;

(d) any orthonormal basis of U can be extended to an orthonormal basis of V.

Proof (continued). For (d), consider any orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U.

(a) U has an orthogonal basis;

(b) any orthogonal basis of U can be extended to an orthogonal basis of V;

(c) U has an orthonormal basis;

(d) any orthonormal basis of U can be extended to an orthonormal basis of V.

Proof (continued). For (d), consider any orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U. In particular, the basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U is orthogonal, and so by (b), it can be extended to an orthogonal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ of V.

(a) U has an orthogonal basis;

(b) any orthogonal basis of U can be extended to an orthogonal basis of V;

(c) U has an orthonormal basis;

(d) any orthonormal basis of U can be extended to an orthonormal basis of V.

Proof (continued). For (d), consider any orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U. In particular, the basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U is orthogonal, and so by (b), it can be extended to an orthogonal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ of V. But then $\{\frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_n}{||\mathbf{u}_n||}\}$ is an orthonormal basis of V. But since the basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U is orthonormal, we know that $||\mathbf{u}_1|| = \dots = ||\mathbf{u}_k|| = 1$, and it follows that $\frac{\mathbf{u}_1}{||\mathbf{u}_1||} = \mathbf{u}_1, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||} = \mathbf{u}_k$.

1. U has an orthogonal basis;
2. any orthogonal basis of U can be extended to an orthogonal basis of V;
3. U has an orthonormal basis;
4. any orthonormal basis of U can be extended to an orthonormal basis of V.

Proof (continued). For (d), consider any orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U. In particular, the basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U is orthogonal, and so by (b), it can be extended to an orthogonal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ of V. But then $\{\frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_n}{||\mathbf{u}_n||}\}$ is an orthonormal basis of V. But since the basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U is orthonormal, we know that $||\mathbf{u}_1|| = \dots = ||\mathbf{u}_k|| = 1$, and it follows that $\frac{\mathbf{u}_1}{||\mathbf{u}_1||} = \mathbf{u}_1, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||} = \mathbf{u}_k$. So, our orthonormal basis $\{\frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_n}{||\mathbf{u}_n||}\}$ of V in fact extends the orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U. This proves (d).

The Gram-Schmidt orthogonalization process (version 2)

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle\cdot,\cdot\rangle$ and the norm $||\cdot||$ induced by $\langle\cdot,\cdot\rangle$. Let $\mathbf{v}_1,\dots,\mathbf{v}_k$ be a linearly independent set of vectors in V. $\forall \ell \in \{1,\dots,k\}$, set

$$\mathbf{u}_\ell = \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \mathrm{proj}_{\mathbf{u}_i}(\mathbf{v}_\ell) = \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \langle \mathbf{v}_\ell, \mathbf{z}_i \rangle \, \mathbf{z}_i;$$

$$\mathbf{z}_\ell = \frac{\mathbf{u}_\ell}{||\mathbf{u}_\ell||}.$$

Then $\{\mathbf{z}_1,\dots,\mathbf{z}_k\}$ is an orthonormal basis of $\mathrm{Span}(\mathbf{v}_1,\dots,\mathbf{v}_k)$.

The Gram-Schmidt orthogonalization process (version 2) recursively creates two sequences of vectors, namely, $\mathbf{u}_1, \ldots, \mathbf{u}_k$ and $\mathbf{z}_1, \ldots, \mathbf{z}_k$, as follows:

- $\mathbf{u}_1 = \mathbf{v}_1$;
- $\mathbf{z}_1 = \frac{\mathbf{u}_1}{||\mathbf{u}_1||}$;
- $\mathbf{u}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{z}_1}(\mathbf{v}_2)$;
- $\mathbf{z}_2 = \frac{\mathbf{u}_2}{||\mathbf{u}_2||}$;
- $\mathbf{u}_3 = \mathbf{v}_3 - \Big(\text{proj}_{\mathbf{z}_1}(\mathbf{v}_3) + \text{proj}_{\mathbf{z}_2}(\mathbf{v}_3)\Big)$;
- $\mathbf{z}_3 = \frac{\mathbf{u}_3}{||\mathbf{u}_3||}$;

$\vdots$

- $\mathbf{u}_k = \mathbf{v}_k - \Big(\text{proj}_{\mathbf{z}_1}(\mathbf{v}_k) + \text{proj}_{\mathbf{z}_2}(\mathbf{v}_k) + \cdots + \text{proj}_{\mathbf{z}_{k-1}}(\mathbf{v}_k)\Big)$;
- $\mathbf{z}_k = \frac{\mathbf{u}_k}{||\mathbf{u}_k||}$.

Example 2.7

Consider the vectors following vectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix}.$$

in $\mathbb{R}^4$, and set $U := \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Using the Gram-Schmidt orthogonalization process (version 2), find an orthonormal basis of U (with respect to the standard scalar product $\cdot$ in $\mathbb{R}^4$ and the norm $||\cdot||$ induced by it).

Solution. See the Lecture Notes.

Definition

Let V be a vector space V over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$.

- For a vector $\mathbf{v} \in V$ and a set of vectors $A \subseteq V$,[a] we say that $\mathbf{v}$ is *orthogonal to* A, and we write $\mathbf{v} \perp A$, provided that $\mathbf{v}$ is orthogonal to all vectors in A.[b]
- For a set $A \subseteq V$, the *orthogonal complement* of A, denoted by $A^{\perp}$, is the set of all vectors in V that are orthogonal to A, i.e.

$$\begin{aligned} A^{\perp} &= \{\mathbf{v} \in V \mid \mathbf{v} \perp A\} \\ &= \{\mathbf{v} \in V \mid \mathbf{v} \perp \mathbf{a} \ \forall \mathbf{a} \in A\} \\ &= \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{a} \rangle = 0 \ \forall \mathbf{a} \in A\}. \end{aligned}$$

[a] A may, but need not be, a subspace of V.

[b] By definition, this means that for all $\mathbf{a} \in A$, we have that $\langle \mathbf{v}, \mathbf{a} \rangle = 0$.

Proposition 3.1

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $A, B \subseteq V$. Then

(a) $A^{\perp}$ is a subspace of V;

(b) if $A \subseteq B$, then $A^{\perp} \supseteq B^{\perp}$.

Proof (outline).

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Proof (outline). For (a), check that $\mathbf{0} \in A^{\perp}$, and that A is closed under vector addition and scalar multiplication (details: Lecture Notes).

.

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Proof (outline). For (a), check that $\mathbf{0} \in A^\perp$, and that A is closed under vector addition and scalar multiplication (details: Lecture Notes).

For (b), suppose that $A \subseteq B$. WTS $A^\perp \supseteq B^\perp$.

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Proof (outline). For (a), check that $\mathbf{0} \in A^{\perp}$, and that A is closed under vector addition and scalar multiplication (details: Lecture Notes).

For (b), suppose that $A \subseteq B$. WTS $A^{\perp} \supseteq B^{\perp}$. Fix $\mathbf{x} \in B^{\perp}$; WTS $\mathbf{x} \in A^{\perp}$.

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.

Proposition 3.1

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $A, B \subseteq V$. Then

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Proof (outline). For (a), check that $\mathbf{0} \in A^{\perp}$, and that A is closed under vector addition and scalar multiplication (details: Lecture Notes).

For (b), suppose that $A \subseteq B$. WTS $A^{\perp} \supseteq B^{\perp}$. Fix $\mathbf{x} \in B^{\perp}$; WTS $\mathbf{x} \in A^{\perp}$. Fix $\mathbf{a} \in A$. Then $\mathbf{a} \in B$ (because $A \subseteq B$), and so $\mathbf{x} \perp \mathbf{a}$ (because $\mathbf{x} \in B^{\perp}$).

Proposition 3.1

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $A, B \subseteq V$. Then

(a) $A^\perp$ is a subspace of V;

(b) if $A \subseteq B$, then $A^\perp \supseteq B^\perp$.

Proof (outline). For (a), check that $\mathbf{0} \in A^\perp$, and that A is closed under vector addition and scalar multiplication (details: Lecture Notes).

For (b), suppose that $A \subseteq B$. WTS $A^\perp \supseteq B^\perp$. Fix $\mathbf{x} \in B^\perp$; WTS $\mathbf{x} \in A^\perp$. Fix $\mathbf{a} \in A$. Then $\mathbf{a} \in B$ (because $A \subseteq B$), and so $\mathbf{x} \perp \mathbf{a}$ (because $\mathbf{x} \in B^\perp$). So, $\mathbf{x} \in A^\perp$. This proves that $A^\perp \supseteq B^\perp$.

Proposition 3.2

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\mathbf{u}_1, \ldots, \mathbf{u}_k \in V$. Then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}^\perp = \mathrm{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k)^\perp$.

Proof.

Proposition 3.2

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\mathbf{u}_1, \dots, \mathbf{u}_k \in V$. Then $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}^\perp = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)^\perp$.

Proof. Since $\{\mathbf{u}_1, \dots, \mathbf{u}_k\} \subseteq \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$, Proposition 3.1(b) guarantees that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}^\perp \supseteq \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)^\perp$.

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Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\mathbf{u}_1, \dots, \mathbf{u}_k \in V$. Then $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}^{\perp} = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)^{\perp}$.

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Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\mathbf{u}_1, \dots, \mathbf{u}_k \in V$. Then $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}^\perp = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)^\perp$.

Proof. Since $\{\mathbf{u}_1, \dots, \mathbf{u}_k\} \subseteq \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$, Proposition 3.1(b) guarantees that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}^\perp \supseteq \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)^\perp$. Let us prove the reverse inclusion. Fix $\mathbf{x} \in \{\mathbf{u}_1, \dots, \mathbf{u}_k\}^\perp$. WTS $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)^\perp$.

Proposition 3.2

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\mathbf{u}_1, \dots, \mathbf{u}_k \in V$. Then $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}^\perp = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)^\perp$.

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Proposition 3.2

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\mathbf{u}_1, \dots, \mathbf{u}_k \in V$. Then $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}^\perp = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)^\perp$.

Proof. Since $\{\mathbf{u}_1, \dots, \mathbf{u}_k\} \subseteq \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$, Proposition 3.1(b) guarantees that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}^\perp \supseteq \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)^\perp$. Let us prove the reverse inclusion. Fix $\mathbf{x} \in \{\mathbf{u}_1, \dots, \mathbf{u}_k\}^\perp$. WTS $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)^\perp$. Fix $\mathbf{u} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then there exist scalars $\alpha_1, \dots, \alpha_k$ such that $\mathbf{u} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k$.

Proposition 3.2

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\mathbf{u}_1, \dots, \mathbf{u}_k \in V$. Then $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}^\perp = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)^\perp$.

Proof. Since $\{\mathbf{u}_1, \dots, \mathbf{u}_k\} \subseteq \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$, Proposition 3.1(b) guarantees that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}^\perp \supseteq \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)^\perp$. Let us prove the reverse inclusion. Fix $\mathbf{x} \in \{\mathbf{u}_1, \dots, \mathbf{u}_k\}^\perp$. WTS $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)^\perp$. Fix $\mathbf{u} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then there exist scalars $\alpha_1, \dots, \alpha_k$ such that $\mathbf{u} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k$. But now

$$
\begin{array}{rcl}
\langle \mathbf{u}, \mathbf{x} \rangle & = & \langle \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k, \mathbf{x} \rangle \\
& = & \alpha_1 \langle \mathbf{u}_1, \mathbf{x} \rangle + \dots + \alpha_k \langle \mathbf{u}_k, \mathbf{x} \rangle \\
& \stackrel{(*)}{=} & \alpha_1 0 + \dots + \alpha_k 0 \\
& = & 0,
\end{array}
$$

where (*) follows from the fact that $\mathbf{x} \in \{\mathbf{u}_1, \dots, \mathbf{u}_k\}^\perp$.

Proposition 3.2

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\mathbf{u}_1, \dots, \mathbf{u}_k \in V$. Then $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}^\perp = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)^\perp$.

Proof. Since $\{\mathbf{u}_1, \dots, \mathbf{u}_k\} \subseteq \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$, Proposition 3.1(b) guarantees that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}^\perp \supseteq \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)^\perp$. Let us prove the reverse inclusion. Fix $\mathbf{x} \in \{\mathbf{u}_1, \dots, \mathbf{u}_k\}^\perp$. WTS $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)^\perp$. Fix $\mathbf{u} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then there exist scalars $\alpha_1, \dots, \alpha_k$ such that $\mathbf{u} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k$. But now

$$
\begin{array}{rcl}
\langle \mathbf{u}, \mathbf{x} \rangle & = & \langle \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k, \mathbf{x} \rangle \\
& = & \alpha_1 \langle \mathbf{u}_1, \mathbf{x} \rangle + \dots + \alpha_k \langle \mathbf{u}_k, \mathbf{x} \rangle \\
& \overset{(*)}{=} & \alpha_1 0 + \dots + \alpha_k 0 \\
& = & 0,
\end{array}
$$

where (*) follows from the fact that $\mathbf{x} \in \{\mathbf{u}_1, \dots, \mathbf{u}_k\}^\perp$. This proves that $\mathbf{x} \perp \mathbf{u}$, and consequently, $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)^\perp$.

Theorem 3.3

Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the norm $|| \cdot ||$ induced by $\langle \cdot, \cdot \rangle$. Let U be a subspace of V. Then all the following hold:

(a) if $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$;

(b) if $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthonormal basis of U, and $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an extension of that basis to an orthonormal basis of V, then $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthonormal basis of $U^\perp$;

(c) $\dim(U) + \dim(U^\perp) = \dim(V)$;

(d) $V = U + U^\perp$;

(e) $(U^\perp)^\perp = U$;

(f) $U \cap U^\perp = \{\mathbf{0}\}$.

(a) If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^{\perp}$.

Proof of (a).

(a) If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

Proof of (a). Assume that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U, and that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V.

(a) If $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

Proof of (a). Assume that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of U, and that $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V. (This implies that $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is linearly independent, and consequently, $\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n$ are all non-zero.)

(a) If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^{\perp}$.

Proof of (a). Assume that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U, and that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V. (This implies that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is linearly independent, and consequently, $\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n$ are all non-zero.) WTS $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^{\perp}$.

(a) If $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

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(a) If $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

Proof of (a). Assume that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of U, and that $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V. (This implies that $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is linearly independent, and consequently, $\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n$ are all non-zero.) WTS $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$. Clearly, $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal set of vectors, and so it suffices to show that $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is in fact a basis of $U^\perp$. By Proposition 2.3, $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is linearly independent, and so we need only show that $\text{Span}(\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}) = U^\perp$.

(a) If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

Proof of (a) (continued). WTS $\text{Span}(\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}) = U^\perp$.

(a) If $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

Proof of (a) (continued). WTS $\text{Span}(\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}) = U^\perp$.

WTS $\text{Span}(\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}) \supseteq U^\perp$.

(a) If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

Proof of (a) (continued). WTS $\text{Span}(\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}) = U^\perp$.

WTS $\text{Span}(\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}) \supseteq U^\perp$. Fix $\mathbf{x} \in U^\perp$.

(a) If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

Proof of (a) (continued). WTS $\text{Span}(\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}) = U^\perp$.

WTS $\text{Span}(\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}) \supseteq U^\perp$. Fix $\mathbf{x} \in U^\perp$. Then $\mathbf{x} \in V$, and so since $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of V, Theorem 2.1 guarantees that

$$\mathbf{x} = \sum_{i=1}^{n} \frac{\langle \mathbf{x}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i.$$

(a) If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

Proof of (a) (continued). WTS $\text{Span}(\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}) = U^\perp$.

WTS $\text{Span}(\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}) \supseteq U^\perp$. Fix $\mathbf{x} \in U^\perp$. Then $\mathbf{x} \in V$, and so since $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of V, Theorem 2.1 guarantees that

$$\mathbf{x} = \sum_{i=1}^{n} \frac{\langle \mathbf{x}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i.$$

Since $\mathbf{x} \in U^\perp$, and since $\mathbf{u}_1, \ldots, \mathbf{u}_k \in U$, we know that $\langle \mathbf{x}, \mathbf{u}_i \rangle = 0$ for all $i \in \{1, \ldots, k\}$.

(a) If $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

Proof of (a) (continued). WTS $\text{Span}(\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}) = U^\perp$.

WTS $\text{Span}(\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}) \supseteq U^\perp$. Fix $\mathbf{x} \in U^\perp$. Then $\mathbf{x} \in V$, and so since $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is an orthogonal basis of V, Theorem 2.1 guarantees that

$$\mathbf{x} = \sum_{i=1}^{n} \frac{\langle \mathbf{x}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i.$$

Since $\mathbf{x} \in U^\perp$, and since $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, we know that $\langle \mathbf{x}, \mathbf{u}_i \rangle = 0$ for all $i \in \{1, \dots, k\}$. Consequently,

$$\mathbf{x} = \sum_{i=k+1}^{n} \frac{\langle \mathbf{x}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i.$$

(a) If $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

Proof of (a) (continued). WTS $\text{Span}(\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}) = U^\perp$.

WTS $\text{Span}(\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}) \supseteq U^\perp$. Fix $\mathbf{x} \in U^\perp$. Then $\mathbf{x} \in V$, and so since $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is an orthogonal basis of V, Theorem 2.1 guarantees that

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Since $\mathbf{x} \in U^\perp$, and since $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, we know that $\langle \mathbf{x}, \mathbf{u}_i \rangle = 0$ for all $i \in \{1, \dots, k\}$. Consequently,

$$\mathbf{x} = \sum_{i=k+1}^{n} \frac{\langle \mathbf{x}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i.$$

Thus, $\mathbf{x}$ is a linear combination of the vectors $\mathbf{u}_{k+1}, \dots, \mathbf{u}_n$, and so $\mathbf{x} \in \text{Span}(\mathbf{u}_{k+1}, \dots, \mathbf{u}_n)$.

(a) If $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

Proof of (a) (continued). WTS $\text{Span}(\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}) = U^\perp$.

WTS $\text{Span}(\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}) \supseteq U^\perp$. Fix $\mathbf{x} \in U^\perp$. Then $\mathbf{x} \in V$, and so since $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is an orthogonal basis of V, Theorem 2.1 guarantees that

$$\mathbf{x} = \sum_{i=1}^{n} \frac{\langle \mathbf{x}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i.$$

Since $\mathbf{x} \in U^\perp$, and since $\mathbf{u}_1, \dots, \mathbf{u}_k \in U$, we know that $\langle \mathbf{x}, \mathbf{u}_i \rangle = 0$ for all $i \in \{1, \dots, k\}$. Consequently,

$$\mathbf{x} = \sum_{i=k+1}^{n} \frac{\langle \mathbf{x}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i.$$

Thus, $\mathbf{x}$ is a linear combination of the vectors $\mathbf{u}_{k+1}, \dots, \mathbf{u}_n$, and so $\mathbf{x} \in \text{Span}(\mathbf{u}_{k+1}, \dots, \mathbf{u}_n)$. Thus, $\text{Span}(\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}) \supseteq U^\perp$.

(a) If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

Proof of (a) (continued). WTS $\text{Span}(\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}) = U^\perp$.

WTS $\text{Span}(\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}) \subseteq U^\perp$.

(a) If $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

Proof of (a) (continued). WTS $\text{Span}(\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}) = U^\perp$.

WTS $\text{Span}(\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}) \subseteq U^\perp$. Fix $\mathbf{x} \in \text{Span}(\mathbf{u}_{k+1}, \dots, \mathbf{u}_n)$; WTS $\mathbf{x} \in U^\perp$.

(a) If $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

Proof of (a) (continued). WTS $\text{Span}(\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}) = U^\perp$.

WTS $\text{Span}(\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}) \subseteq U^\perp$. Fix $\mathbf{x} \in \text{Span}(\mathbf{u}_{k+1}, \dots, \mathbf{u}_n)$; WTS $\mathbf{x} \in U^\perp$. Fix scalars $\alpha_{k+1}, \dots, \alpha_n$ such that

$$\mathbf{x} = \alpha_{k+1}\mathbf{u}_{k+1} + \cdots + \alpha_n\mathbf{u}_n.$$

(a) If $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

Proof of (a) (continued). WTS $\text{Span}(\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}) = U^\perp$.

WTS $\text{Span}(\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}) \subseteq U^\perp$. Fix $\mathbf{x} \in \text{Span}(\mathbf{u}_{k+1}, \dots, \mathbf{u}_n)$; WTS $\mathbf{x} \in U^\perp$. Fix scalars $\alpha_{k+1}, \dots, \alpha_n$ such that

$$\mathbf{x} = \alpha_{k+1}\mathbf{u}_{k+1} + \cdots + \alpha_n\mathbf{u}_n.$$

Fix $\mathbf{u} \in U$; WTS $\mathbf{x} \perp \mathbf{u}$.

(a) If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

Proof of (a) (continued). WTS $\text{Span}(\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}) = U^\perp$.

WTS $\text{Span}(\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}) \subseteq U^\perp$. Fix $\mathbf{x} \in \text{Span}(\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n)$; WTS $\mathbf{x} \in U^\perp$. Fix scalars $\alpha_{k+1}, \ldots, \alpha_n$ such that

$$\mathbf{x} = \alpha_{k+1}\mathbf{u}_{k+1} + \cdots + \alpha_n \mathbf{u}_n.$$

Fix $\mathbf{u} \in U$; WTS $\mathbf{x} \perp \mathbf{u}$. Since $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a basis of U, there exist scalars $\alpha_1, \ldots, \alpha_k$ such that $\mathbf{u} = \alpha_1\mathbf{u}_1 + \cdots + \alpha_k\mathbf{u}_k$.

(a) If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

Proof of (a) (continued). WTS $\text{Span}(\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}) = U^\perp$.

WTS $\text{Span}(\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}) \subseteq U^\perp$. Fix $\mathbf{x} \in \text{Span}(\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n)$; WTS $\mathbf{x} \in U^\perp$. Fix scalars $\alpha_{k+1}, \ldots, \alpha_n$ such that

$$\mathbf{x} \quad = \quad \alpha_{k+1}\mathbf{u}_{k+1} + \cdots + \alpha_n\mathbf{u}_n.$$

Fix $\mathbf{u} \in U$; WTS $\mathbf{x} \perp \mathbf{u}$. Since $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a basis of U, there exist scalars $\alpha_1, \ldots, \alpha_k$ such that $\mathbf{u} = \alpha_1\mathbf{u}_1 + \cdots + \alpha_k\mathbf{u}_k$. If V is a vector space over $\mathbb{C}$ (it's similar for $\mathbb{R}$), then we have that

$$\begin{aligned} \langle \mathbf{x}, \mathbf{u} \rangle &= \langle \alpha_{k+1}\mathbf{u}_{k+1} + \cdots + \alpha_n\mathbf{u}_n, \alpha_1\mathbf{u}_1 + \cdots + \alpha_k\mathbf{u}_k \rangle \\ &= \sum_{i=k+1}^{n} \sum_{j=1}^{k} \alpha_i \overline{\alpha_j} \langle \mathbf{u}_i, \mathbf{u}_j \rangle \overset{(*)}{=} 0, \end{aligned}$$

where (*) follows from the orthogonality of $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$. So, $\mathbf{x} \perp \mathbf{u}$, and consequently, $\mathbf{x} \in U^\perp$. This proves (a).

(b) If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormal basis of U, and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an extension of that basis to an orthonormal basis of V, then $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthonormal basis of $U^\perp$.

Proof of (b). Analogous to (a).

(f) $U \cap U^{\perp} = \{\mathbf{0}\}$.

Proof of (f). By hypothesis, U is a subspace of V, and by Proposition 3.1(a), $U^{\perp}$ is also a subspace of V. So, both U and $U^{\perp}$ contain $\mathbf{0}$, i.e. $\{\mathbf{0}\} \subseteq U \cap U^{\perp}$. Now, fix any $\mathbf{u} \in U \cap U^{\perp}$; we must show that $\mathbf{u} = \mathbf{0}$. Since $\mathbf{u} \in U$ and $\mathbf{u} \in U^{\perp}$, we have that $\mathbf{u} \perp \mathbf{u}$, i.e. $\langle \mathbf{u}, \mathbf{u} \rangle = 0$. But then by the definition of a scalar product, we have that $\mathbf{u} = \mathbf{0}$. This proves (f).

(c) $\dim(U) + \dim(U^{\perp}) = \dim(V)$;

(d) $V = U + U^{\perp}$;

(e) $(U^{\perp})^{\perp} = U$.

Proof.

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(d) $V = U + U^{\perp}$;

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Proof. First, since V is finite-dimensional, so is U. So, by Corollary 2.5(a), U has an orthogonal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$. By Corollary 2.5(b), $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ can be extended to an orthogonal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ of V. By (a), $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^{\perp}$.

(c) $\dim(U) + \dim(U^{\perp}) = \dim(V)$.

Proof of (c). Reminder: $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of V; $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U; $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^{\perp}$.

(c) $\dim(U) + \dim(U^{\perp}) = \dim(V)$.

Proof of (c). Reminder: $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of V; $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U; $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^{\perp}$.

Note that

- $\dim(V) = n$;
- $\dim(U) = k$;
- $\dim(U^{\perp}) = n - k$.

(c) $\dim(U) + \dim(U^{\perp}) = \dim(V)$.

Proof of (c). Reminder: $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of V; $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U; $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^{\perp}$.

Note that

- $\dim(V) = n$;
- $\dim(U) = k$;
- $\dim(U^{\perp}) = n - k$.

So, $\dim(U) + \dim(U^{\perp}) = k + (n - k) = n = \dim(V)$. This proves (c).

(d) $V = U + U^{\perp}$.

Proof of (d). Reminder: $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of V; $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U; $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^{\perp}$.

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It is clear that $U + U^{\perp} \subseteq V$, and we need only show that $V \subseteq U + U^{\perp}$.

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It is clear that $U + U^{\perp} \subseteq V$, and we need only show that $V \subseteq U + U^{\perp}$. Fix $\mathbf{v} \in V$.

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It is clear that $U + U^{\perp} \subseteq V$, and we need only show that $V \subseteq U + U^{\perp}$. Fix $\mathbf{v} \in V$. Since $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is a basis of V, we know that there exist scalars $\alpha_1, \dots, \alpha_n$ such that $\mathbf{v} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_n \mathbf{u}_n$.

(d) $V = U + U^{\perp}$.

Proof of (d). Reminder: $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of V; $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of U; $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of $U^{\perp}$.

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Set $\mathbf{v}_1 := \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k$ and $\mathbf{v}_2 := \alpha_{k+1} \mathbf{u}_{k+1} + \dots + \alpha_n \mathbf{u}_n$.

(d) $V = U + U^{\perp}$.

Proof of (d). Reminder: $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of V; $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of U; $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of $U^{\perp}$.

It is clear that $U + U^{\perp} \subseteq V$, and we need only show that $V \subseteq U + U^{\perp}$. Fix $\mathbf{v} \in V$. Since $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is a basis of V, we know that there exist scalars $\alpha_1, \dots, \alpha_n$ such that $\mathbf{v} = \alpha_1\mathbf{u}_1 + \dots + \alpha_n\mathbf{u}_n$.

Set $\mathbf{v}_1 := \alpha_1\mathbf{u}_1 + \dots + \alpha_k\mathbf{u}_k$ and $\mathbf{v}_2 := \alpha_{k+1}\mathbf{u}_{k+1} + \dots + \alpha_n\mathbf{u}_n$.

Then $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$.

(d) $V = U + U^{\perp}$.

Proof of (d). Reminder: $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of V; $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of U; $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of $U^{\perp}$.

It is clear that $U + U^{\perp} \subseteq V$, and we need only show that $V \subseteq U + U^{\perp}$. Fix $\mathbf{v} \in V$. Since $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is a basis of V, we know that there exist scalars $\alpha_1, \dots, \alpha_n$ such that $\mathbf{v} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_n \mathbf{u}_n$.

Set $\mathbf{v}_1 := \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k$ and $\mathbf{v}_2 := \alpha_{k+1} \mathbf{u}_{k+1} + \dots + \alpha_n \mathbf{u}_n$.

Then $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. Since $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of U, we see that $\mathbf{v}_1 \in U$, and since $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is a basis of $U^{\perp}$, we see that $\mathbf{v}_2 \in U^{\perp}$.

(d) $V = U + U^\perp$.

Proof of (d). Reminder: $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of V; $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of U; $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

It is clear that $U + U^\perp \subseteq V$, and we need only show that $V \subseteq U + U^\perp$. Fix $\mathbf{v} \in V$. Since $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is a basis of V, we know that there exist scalars $\alpha_1, \dots, \alpha_n$ such that $\mathbf{v} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_n \mathbf{u}_n$.

Set $\mathbf{v}_1 := \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k$ and $\mathbf{v}_2 := \alpha_{k+1} \mathbf{u}_{k+1} + \dots + \alpha_n \mathbf{u}_n$.

Then $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. Since $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of U, we see that $\mathbf{v}_1 \in U$, and since $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is a basis of $U^\perp$, we see that $\mathbf{v}_2 \in U^\perp$. So, $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ belongs to $U + U^\perp$, and it follows that $V \subseteq U + U^\perp$. This proves (d).

(a) If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

(e) $(U^\perp)^\perp = U$.

Proof of (e). Reminder: $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of V; $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U; $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

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We know that $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$, and that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of V extending $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$.

(a) If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

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We know that $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$, and that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of V extending $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$. So, by (a), $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of $(U^\perp)^\perp$.

(a) If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

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Proof of (e). Reminder: $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of V; $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U; $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

We know that $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$, and that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of V extending $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$. So, by (a), $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of $(U^\perp)^\perp$. But by construction, $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U.

(a) If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U, and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an extension of that basis to an orthogonal basis of V, then $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

(e) $(U^\perp)^\perp = U$.

Proof of (e). Reminder: $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of V; $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U; $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$.

We know that $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of $U^\perp$, and that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of V extending $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$. So, by (a), $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of $(U^\perp)^\perp$. But by construction, $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of U. So, $U = \mathrm{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k) = (U^\perp)^\perp$. This proves (e).