Linear Algebra 2

Lecture #11

Scalar products and norms. The Cauchy-Schwarz inequality

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February 16, 2023

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- As a trade-off for imposing this additional structure, we restrict ourselves to vector spaces over only two fields: $\mathbb{R}$ and $\mathbb{C}$.
  - The theory that we develop in this lecture would not work for vector spaces over general fields $\mathbb{F}$.

- Scalar products in real vector spaces

Definition

A *scalar product* (also called *inner product*) in a vector space V over the field $\mathbb{R}$ is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies the following axioms:

r.1. for all $\mathbf{x} \in V$, $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, and equality holds iff $\mathbf{x} = \mathbf{0}$;

r.2. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$;

r.3. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{R}$, $\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$;

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- Axioms r.2 and r.3 guarantee that the scalar product in a vector space V over $\mathbb{R}$ is linear in the first variable (when we keep the second variable fixed).
- But in fact, axioms r.2, r.3, and r.4 guarantee that it is linear in the second variable as well (when we keep the first variable fixed). Indeed (next slide):

r.2'. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, we have that

$$\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle \quad = \quad \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle,$$

since $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle \overset{r.4}{=} \langle \mathbf{y} + \mathbf{z}, \mathbf{x} \rangle \overset{r.2}{=} \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{z}, \mathbf{x} \rangle \overset{r.4}{=} \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$;

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$$\langle \mathbf{x}, \alpha \mathbf{y} \rangle \quad = \quad \alpha \langle \mathbf{x}, \mathbf{y} \rangle,$$

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r.3. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{R}$, $\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$;

r.4. for all $\mathbf{x}, \mathbf{y} \in V$, $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.

As a corollary of the above axioms, we also got:

r.2'. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$;

r.3'. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{R}$, $\langle \mathbf{x}, \alpha\mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$.

Definition

Given vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$ in $\mathbb{R}^n$, the *standard scalar product* of $\mathbf{x}$ and $\mathbf{y}$, denoted by $\mathbf{x} \cdot \mathbf{y}$, is given by

$$\mathbf{x} \cdot \mathbf{y} \quad := \quad \sum_{i=1}^{n} x_i y_i$$

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- For example, for vectors $\begin{bmatrix} 1 & -2 & 5 \end{bmatrix}^T, \begin{bmatrix} -3 & 2 & 1 \end{bmatrix}^T \in \mathbb{R}^3$, we have

$$\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \quad = \quad 1 \cdot (-3) + (-2) \cdot 2 + 5 \cdot 1 \quad = \quad -2.$$

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- But is the standard scalar product in $\mathbb{R}^n$ an actual scalar product? That is, does the standard scalar product satisfy axioms r.1-r.4?

Proposition 1.1

The standard scalar product in $\mathbb{R}^n$ is a scalar product.

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r.1. For a vector $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ in $\mathbb{R}^n$, we have that

$$\mathbf{x} \cdot \mathbf{x} \;=\; \sum_{i=1}^{n} x_i^2 \;\overset{(*)}{\geq}\; 0,$$

and (*) is an equality iff $x_1 = \cdots = x_n = 0$, i.e. iff $\mathbf{x} = \mathbf{0}$.

Proposition 1.1

The standard scalar product in $\mathbb{R}^n$ is a scalar product.

Proof (continued).

r.2. For vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$, $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$, and $\mathbf{z} = \begin{bmatrix} z_1 & \dots & z_n \end{bmatrix}^T$ in $\mathbb{R}^n$, we have that

$$\begin{aligned} (\mathbf{x}+\mathbf{y}) \cdot \mathbf{z} &= \sum_{i=1}^{n} (x_i + y_i) z_i \\ &= \Big(\sum_{i=1}^{n} x_i z_i \Big) + \Big(\sum_{i=1}^{n} y_i z_i \Big) \\ &= \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z}. \end{aligned}$$

Proposition 1.1

The standard scalar product in $\mathbb{R}^n$ is a scalar product.

Proof (continued).

r.3. For vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$ in $\mathbb{R}^n$ and a scalar $\alpha \in \mathbb{R}$, we have that

$$(\alpha \mathbf{x}) \cdot \mathbf{y} \;=\; \sum_{i=1}^{n} (\alpha x_i) y_i \;=\; \alpha \sum_{i=1}^{n} x_i y_i \;=\; \alpha(\mathbf{x} \cdot \mathbf{y}).$$

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$$\mathbf{x} \cdot \mathbf{y} \;=\; \sum_{i=1}^{n} x_i y_i \;=\; \sum_{i=1}^{n} y_i x_i \;=\; \mathbf{y} \cdot \mathbf{x}.$$

Q.E.D.

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$$\mathbf{x}^T\mathbf{y} \quad = \quad \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad = \quad \begin{bmatrix} \sum_{i=1}^{n} x_i y_i \end{bmatrix} \quad = \quad \begin{bmatrix} \mathbf{x} \cdot \mathbf{y} \end{bmatrix}$$

- So, if we identify 1×1 matrices with scalars, then we simply get that

$$\mathbf{x} \cdot \mathbf{y} \quad = \quad \mathbf{x}^T\mathbf{y}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Definition

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- A similar type of scalar product can be defined for matrices. Indeed, for matrices $A = [a_{i,j}]_{n \times m}$ and $B = [b_{i,j}]_{n \times m}$ in $\mathbb{R}^{n \times m}$, we can define

$$\langle A, B \rangle \quad = \quad \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} b_{ij}.$$

- It is easy to verify that this really is a scalar product in $\mathbb{R}^{n \times m}$ (the proof is similar to that of Proposition 1.1).

- For those who have studied calculus:

Proposition 1.2

Let $a, b \in \mathbb{R}$ be such that $a < b$, and let $\mathcal{C}_{[a,b]}$ be the vector space (over $\mathbb{R}$) of all continuous functions from the closed interval $[a, b]$ to $\mathbb{R}$.[a] Then the function $\langle \cdot, \cdot \rangle : \mathcal{C}_{[a,b]} \times \mathcal{C}_{[a,b]} \to \mathbb{R}$ defined by

$$\langle f, g \rangle \quad := \quad \int_a^b f(x)g(x)dx$$

for all $f, g \in \mathcal{C}_{[a,b]}$ is a scalar product.

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[a] Recall from calculus that all such functions are integrable.

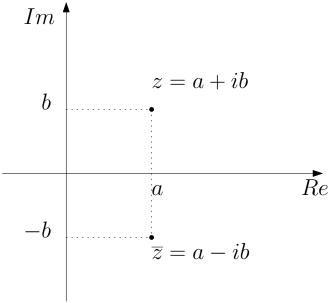
Proof. Lecture Notes.

- A quick review of complex numbers

Definition

Given a complex number $z = a + ib$ (where $a, b \in \mathbb{R}$),

- the *real part* of z is the real number $\text{Re}(z) = a$;
- the *imaginary part* of z is the real number $\text{Im}(z) = b$;
- the *absolute value* of z is the real number $|z| = \sqrt{a^2 + b^2}$;
- the *complex conjugate* of z is the complex number $\overline{z} = a - ib$.



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Remark 1.3

For a complex number $z = a + ib$ (where $a, b \in \mathbb{R}$), we have that

$$z\overline{z} \;=\; (a+ib)(a-ib) \;=\; a^2 - i^2b^2 \;=\; a^2 + b^2 \;=\; |z|^2;$$

in particular, $z\overline{z} \geq 0$, and equality holds iff $z = 0$.

Proposition 1.4

For all $z_1, z_2 \in \mathbb{C}$, the following hold:

1. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$;
2. $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$;
3. $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$;
4. if $z_2 \neq 0$, then $\overline{z_1/z_2} = \overline{z_1}/\overline{z_2}$.

Moreover, for all $z \in \mathbb{C}$ and non-negative integers n, we have that

5. $\overline{z^n} = (\overline{z})^n$.

Proof. High school math (or any introductory text on complex numbers).

- Scalar products in complex vector spaces

Definition

A *scalar product* (also called *inner product*) in a vector space V over the field $\mathbb{C}$ is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ that satisfies the following axioms:

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c.2. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$;

c.3. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{C}$, $\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$;

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- Axioms c.2 and c.3 guarantee that the scalar product in a vector space V over $\mathbb{C}$ is linear in the first variable (when we keep the second variable fixed).

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Definition

A *scalar product* (also called *inner product*) in a vector space V over the field $\mathbb{C}$ is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ that satisfies the following axioms:

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c.2. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$;

c.3. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{C}$, $\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$;

c.4. for all $\mathbf{x}, \mathbf{y} \in V$, $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$.

- Axioms c.2 and c.3 guarantee that the scalar product in a vector space V over $\mathbb{C}$ is linear in the first variable (when we keep the second variable fixed).
- Unlike in the real case, it is **not** linear in the second variable (when we keep the first variable fixed).

- Scalar products in complex vector spaces

Definition

A *scalar product* (also called *inner product*) in a vector space V over the field $\mathbb{C}$ is a function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ that satisfies the following axioms:

c.1. for all $\mathbf{x} \in V$, $\langle\mathbf{x},\mathbf{x}\rangle$ is a real number, $\langle\mathbf{x},\mathbf{x}\rangle \geq 0$, and equality holds iff $\mathbf{x} = \mathbf{0}$;

c.2. for all $\mathbf{x},\mathbf{y},\mathbf{z} \in V$, $\langle\mathbf{x}+\mathbf{y},\mathbf{z}\rangle = \langle\mathbf{x},\mathbf{z}\rangle + \langle\mathbf{y},\mathbf{z}\rangle$;

c.3. for all $\mathbf{x},\mathbf{y} \in V$ and $\alpha \in \mathbb{C}$, $\langle\alpha\mathbf{x},\mathbf{y}\rangle = \alpha\langle\mathbf{x},\mathbf{y}\rangle$;

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- Unlike in the real case, it is **not** linear in the second variable (when we keep the first variable fixed).
- We do, however, have the following (next slide):

c.2'. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, we have that

$$\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle \quad = \quad \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle,$$

since $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle \overset{c.4}{=} \overline{\langle \mathbf{y} + \mathbf{z}, \mathbf{x} \rangle} \overset{c.2}{=} \overline{\langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{z}, \mathbf{x} \rangle} =$
$\overline{\langle \mathbf{y}, \mathbf{x} \rangle} + \overline{\langle \mathbf{z}, \mathbf{x} \rangle} \overset{c.4}{=} \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$;

c.2'. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, we have that

$$\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle \quad = \quad \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle,$$

since $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle \overset{c.4}{=} \overline{\langle \mathbf{y} + \mathbf{z}, \mathbf{x} \rangle} \overset{c.2}{=} \overline{\langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{z}, \mathbf{x} \rangle} =$
$\overline{\langle \mathbf{y}, \mathbf{x} \rangle} + \overline{\langle \mathbf{z}, \mathbf{x} \rangle} \overset{c.4}{=} \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$;

c.3'. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{C}$, we have that

$$\langle \mathbf{x}, \alpha \mathbf{y} \rangle \quad = \quad \overline{\alpha} \langle \mathbf{x}, \mathbf{y} \rangle,$$

since $\langle \mathbf{x}, \alpha \mathbf{y} \rangle \overset{c.4}{=} \overline{\langle \alpha \mathbf{y}, \mathbf{x} \rangle} = \overline{\alpha} \overline{\langle \mathbf{y}, \mathbf{x} \rangle} \overset{c.4}{=} \overline{\alpha} \langle \mathbf{x}, \mathbf{y} \rangle$.

Definition

A *scalar product* (also called *inner product*) in a vector space V over the field $\mathbb{C}$ is a function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ that satisfies the following axioms:

c.1. for all $\mathbf{x} \in V$, $\langle \mathbf{x}, \mathbf{x}\rangle$ is a real number, $\langle \mathbf{x}, \mathbf{x}\rangle \geq 0$, and equality holds iff $\mathbf{x} = \mathbf{0}$;

c.2. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x} + \mathbf{y}, \mathbf{z}\rangle = \langle \mathbf{x}, \mathbf{z}\rangle + \langle \mathbf{y}, \mathbf{z}\rangle$;

c.3. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{R}$, $\langle \alpha\mathbf{x}, \mathbf{y}\rangle = \alpha\langle \mathbf{x}, \mathbf{y}\rangle$;

c.4. for all $\mathbf{x}, \mathbf{y} \in V$, $\langle \mathbf{x}, \mathbf{y}\rangle = \overline{\langle \mathbf{y}, \mathbf{x}\rangle}$.

As a corollary of the above axioms, we also got:

c.2'. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x}, \mathbf{y} + \mathbf{z}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{x}, \mathbf{z}\rangle$,

c.3'. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{C}$, $\langle \mathbf{x}, \alpha\mathbf{y}\rangle = \overline{\alpha}\langle \mathbf{x}, \mathbf{y}\rangle$,

Definition

Given vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$ in $\mathbb{C}^n$, the *standard scalar product* of $\mathbf{x}$ and $\mathbf{y}$, denoted by $\mathbf{x} \cdot \mathbf{y}$, is given by

$$\mathbf{x} \cdot \mathbf{y} \quad := \quad \sum_{i=1}^{n} x_i \overline{y_i}.$$

- For example, for vectors $\begin{bmatrix} 1-2i & -2+i \end{bmatrix}^T$ and $\begin{bmatrix} 2+i & 1+3i \end{bmatrix}^T$ in $\mathbb{C}^2$, we have

$$\begin{aligned} \begin{bmatrix} 1-2i \\ -2+i \end{bmatrix} \cdot \begin{bmatrix} 2+i \\ 1+3i \end{bmatrix} &= (1-2i)\overline{(2+i)} + (-2+i)\overline{(1+3i)} \\ &= (1-2i)(2-i) + (-2+i)(1-3i) \\ &= 1+2i. \end{aligned}$$

Proposition 1.5

The standard scalar product in $\mathbb{C}^n$ is a scalar product.

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c.1. For a vector $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}$ in $\mathbb{C}^n$, we have that

$$\mathbf{x} \cdot \mathbf{x} \;=\; \sum_{i=1}^{n} x_i \overline{x_i} \;\overset{(*)}{\geq}\; 0,$$

where (*) follows from Remark 1.3. Moreover, by Remark 1.3, the inequality (*) is an equality iff $x_1 = \dots = x_n = 0$, i.e. iff $\mathbf{x} = \mathbf{0}$.

Proposition 1.5

The standard scalar product in $\mathbb{C}^n$ is a scalar product.

Proof (continued).

c.2 For vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$, $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$, and $\mathbf{z} = \begin{bmatrix} z_1 & \dots & z_n \end{bmatrix}^T$ in $\mathbb{C}^n$, we have that

$$\begin{aligned} (\mathbf{x}+\mathbf{y})\cdot\mathbf{z} &= \sum_{i=1}^{n} (x_i + y_i)\overline{z_i} \\ &= \Big(\sum_{i=1}^{n} x_i\overline{z_i}\Big) + \Big(\sum_{i=1}^{n} y_i\overline{z_i}\Big) \\ &= \mathbf{x}\cdot\mathbf{z} + \mathbf{y}\cdot\mathbf{z}. \end{aligned}$$

Proposition 1.5

The standard scalar product in $\mathbb{C}^n$ is a scalar product.

Proof (continued).

c.3. For vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$ in $\mathbb{C}^n$ and a scalar $\alpha \in \mathbb{C}$, we have that

$$(\alpha\mathbf{x}) \cdot \mathbf{y} \;=\; \sum_{i=1}^{n} (\alpha x_i)\overline{y_i} \;=\; \alpha \sum_{i=1}^{n} x_i \overline{y_i} \;=\; \alpha(\mathbf{x} \cdot \mathbf{y}).$$

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$$\mathbf{x} \cdot \mathbf{y} \quad = \quad \sum_{i=1}^{n} x_i\overline{y_i} \quad = \quad \sum_{i=1}^{n} \overline{\overline{x_i} y_i} \quad = \quad \overline{\sum_{i=1}^{n} \overline{x_i} y_i} \quad = \quad \overline{\sum_{i=1}^{n} y_i \overline{x_i}} \quad = \quad \overline{\mathbf{y} \cdot \mathbf{x}}.$$

Q.E.D.

Definition

Given a scalar product $\langle \cdot, \cdot \rangle$ in a vector space V over $\mathbb{R}$ or $\mathbb{C}$, we define the *norm in V induced by* $\langle \cdot, \cdot \rangle$ to be the function $|| \cdot || : V \to \mathbb{R}$ given by

$$||\mathbf{x}|| \quad := \quad \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$$

for all $\mathbf{x} \in V$.

- Remark: In view of r.1 and c.1, for all $\mathbf{x} \in V$, we have that $||\mathbf{x}||$ is a non-negative **real** number (even if V is a vector space over $\mathbb{C}$), and moreover, $||\mathbf{x}|| = 0$ iff $\mathbf{x} = \mathbf{0}$.

Proposition 2.1

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, let $\langle \cdot, \cdot \rangle$ be a scalar product in V, and let $|| \cdot ||$ be the norm in V induced by $\langle \cdot, \cdot \rangle$. Then for all vectors $\mathbf{x} \in V$ and scalars α, we have that

$$||\alpha \mathbf{x}|| \quad = \quad |\alpha| \; ||\mathbf{x}||.$$

Proof.

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$$||\alpha\mathbf{x}|| \quad = \quad |\alpha|\ ||\mathbf{x}||.$$

Proof. If the vector space V is over $\mathbb{R}$, then for all vectors $\mathbf{x} \in V$ and scalars $\alpha \in \mathbb{R}$, we have that

$$\begin{aligned} ||\alpha\mathbf{x}|| &= \sqrt{\langle\alpha\mathbf{x},\alpha\mathbf{x}\rangle} \\ &= \sqrt{\alpha^2\langle\mathbf{x},\mathbf{x}\rangle} && \text{by r.3 and r.3'} \\ &= |\alpha|\sqrt{\langle\mathbf{x},\mathbf{x}\rangle} \\ &= |\alpha|\ ||\mathbf{x}||. \end{aligned}$$

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Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, let $\langle\cdot,\cdot\rangle$ be a scalar product in V, and let $||\cdot||$ be the norm in V induced by $\langle\cdot,\cdot\rangle$. Then for all vectors $\mathbf{x} \in V$ and scalars α, we have that

$$||\alpha\mathbf{x}|| \quad = \quad |\alpha|\ ||\mathbf{x}||.$$

Proof. On the other hand, if the vector space V is over $\mathbb{C}$, then for all vectors $\mathbf{x} \in V$ and scalars $\alpha \in \mathbb{C}$, we have that

$$\begin{aligned}
||\alpha\mathbf{x}|| &= \sqrt{\langle\alpha\mathbf{x},\alpha\mathbf{x}\rangle} \\
&= \sqrt{\alpha\overline{\alpha}\langle\mathbf{x},\mathbf{x}\rangle} && \text{by c.3 and c.3'} \\
&= \sqrt{|\alpha|^2\langle\mathbf{x},\mathbf{x}\rangle} && \text{by Remark 1.3} \\
&= |\alpha|\sqrt{\langle\mathbf{x},\mathbf{x}\rangle} \\
&= |\alpha|\ ||\mathbf{x}||. && \text{Q.E.D.}
\end{aligned}$$

- Note that if $||\cdot||$ is the norm induced on $\mathbb{R}^n$ by the standard scalar product in $\mathbb{R}^n$, then for all $\mathbf{x} = \left[\begin{array}{ccc} x_1 & \dots & x_n \end{array}\right]^T$, we have that

$$||\mathbf{x}|| \;=\; \sqrt{\mathbf{x}\cdot\mathbf{x}} \;=\; \sqrt{\sum_{i=1}^{n} x_i^2}.$$

  - So, we simply get the standard Euclidean length in $\mathbb{R}^n$.

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$$\mathbf{x}\cdot\mathbf{y} \;=\; ||\mathbf{x}||\;||\mathbf{y}||\;\cos\theta,$$

where θ is the angle between **x** and **y**.

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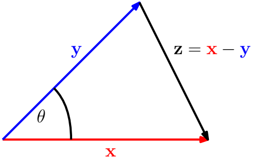
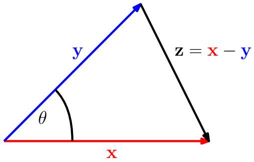
where θ is the angle between **x** and **y**.
  - Let's check this!

- Consider the (possibly degenerate) triangle formed by $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z} := \mathbf{x} - \mathbf{y}$, and let θ be the angle between $\mathbf{x}$ and $\mathbf{y}$ in this triangle.

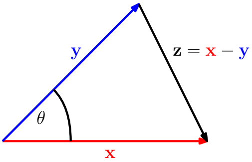


- We then compute

$$\begin{aligned} ||\mathbf{z}||^2 &= \mathbf{z} \cdot \mathbf{z} \\ &= (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \\ &= \underbrace{\mathbf{x} \cdot \mathbf{x}}_{=||\mathbf{x}||^2} - \mathbf{x} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{x} + \underbrace{\mathbf{y} \cdot \mathbf{y}}_{=||\mathbf{y}||^2} \\ &= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2\mathbf{x} \cdot \mathbf{y} \end{aligned}$$



- Reminder: $||\mathbf{z}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2\mathbf{x} \cdot \mathbf{y}$



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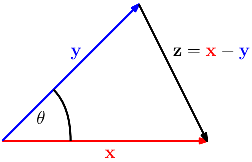
$$||\mathbf{z}||^2 \quad = \quad ||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2||\mathbf{x}||\ ||\mathbf{y}||\ \cos\theta.$$



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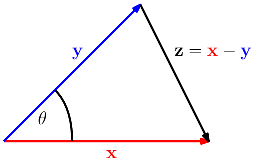
- So, $||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2\mathbf{x}\cdot\mathbf{y} = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2||\mathbf{x}||\ ||\mathbf{y}||\ \cos\theta$, and consequently,

$$\mathbf{x}\cdot\mathbf{y} \quad = \quad ||\mathbf{x}||\ ||\mathbf{y}||\ \cos\theta,$$

as we had claimed.



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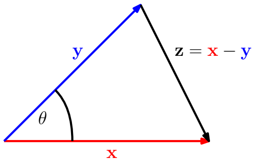
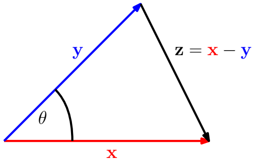


- Reminder: $\mathbf{x} \cdot \mathbf{y} = ||\mathbf{x}|| \, ||\mathbf{y}|| \, \cos\theta$, where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$.

- This means that non-zero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are orthogonal (i.e. the angle between them is 90°) iff $\mathbf{x} \cdot \mathbf{y} = 0$.
  - This is because for an angle θ, with $0^\circ \leq \theta \leq 180^\circ$, we have that $\cos\theta = 0$ iff $\theta = 90^\circ$.



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- Let's now return to norms defined by arbitrary scalar products.

Definition

Given a vector space V over $\mathbb{R}$ or $\mathbb{C}$, and given a scalar product $\langle\cdot,\cdot\rangle$ on V, we say that vectors $\mathbf{x}$ and $\mathbf{y}$ in V are *orthogonal* (and we write $\mathbf{x} \perp \mathbf{y}$) if $\langle\mathbf{x},\mathbf{y}\rangle = 0$.

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  - For example, for the scalar product defined on $\mathcal{C}_{[-\pi,\pi]}$ in Proposition 1.2, we have that $\sin x \perp \cos x$, since $\langle \sin x, \cos x \rangle = \int\limits_{-\pi}^{\pi} \sin x \cos x dx = 0$.

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- It follows from r.4 and c.4 that $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ iff $\langle \mathbf{y}, \mathbf{x} \rangle = 0$.
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  - So, $\mathbf{x} \perp \mathbf{y}$ iff $\mathbf{y} \perp \mathbf{x}$, as one would expect.
- Also, $\mathbf{0}$ is orthogonal to every vector in V.
  - Indeed, for all $\mathbf{y} \in V$, $\langle \mathbf{0}, \mathbf{y} \rangle = \langle 0\mathbf{0}, \mathbf{y} \rangle \overset{(*)}{=} 0 \langle \mathbf{0}, \mathbf{y} \rangle = 0$, where (*) follows from r.3 or c.3.

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The Pythagorean theorem

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, let $\langle\cdot,\cdot\rangle$ be a scalar product in V, and let $||\cdot||$ be the norm in V induced by the scalar product $\langle\cdot,\cdot\rangle$. Let $\mathbf{x}$ and $\mathbf{y}$ be orthogonal vectors in V. Then

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$$||\mathbf{x}+\mathbf{y}||^2 \quad = \quad ||\mathbf{x}||^2 + ||\mathbf{y}||^2.$$

Proof. Since $\mathbf{x} \perp \mathbf{y}$, we have that $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ and $\langle \mathbf{y}, \mathbf{x} \rangle = 0$. So,

$$\begin{aligned} ||\mathbf{x}+\mathbf{y}||^2 &= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle \\ &= \underbrace{\langle \mathbf{x}, \mathbf{x} \rangle}_{=||\mathbf{x}||^2} + \underbrace{\langle \mathbf{x}, \mathbf{y} \rangle}_{=0} + \underbrace{\langle \mathbf{y}, \mathbf{x} \rangle}_{=0} + \underbrace{\langle \mathbf{y}, \mathbf{y} \rangle}_{=||\mathbf{y}||^2} \\ &= ||\mathbf{x}||^2 + ||\mathbf{y}||^2, \end{aligned}$$

which is what we needed to show.

The Cauchy–Schwarz inequality

Let $\langle \cdot, \cdot \rangle$ be a scalar product in a vector space V over $\mathbb{R}$ or $\mathbb{C}$, and let $|| \cdot ||$ be the norm in V induced by $\langle \cdot, \cdot \rangle$. Then

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \quad \leq \quad ||\mathbf{x}|| \, ||\mathbf{y}||$$

for all $\mathbf{x}, \mathbf{y} \in V$.

Proof.

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Proof. Fix $\mathbf{x}, \mathbf{y} \in V$.

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Proof. Fix $\mathbf{x}, \mathbf{y} \in V$. WMA $\langle \mathbf{x}, \mathbf{y} \rangle \neq 0$, for otherwise, the result is immediate.

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Proof. Fix $\mathbf{x}, \mathbf{y} \in V$. WMA $\langle \mathbf{x}, \mathbf{y} \rangle \neq 0$, for otherwise, the result is immediate. This implies that $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$, and consequently, $||\mathbf{x}||, ||\mathbf{y}|| \neq 0$. We set

$$\mathbf{z} \quad := \quad \tfrac{\langle \mathbf{y}, \mathbf{y} \rangle}{\langle \mathbf{x}, \mathbf{y} \rangle} \mathbf{x} - \mathbf{y},$$

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and we compute

$$\langle \mathbf{z}, \mathbf{y} \rangle \quad = \quad \langle \tfrac{\langle \mathbf{y}, \mathbf{y} \rangle}{\langle \mathbf{x}, \mathbf{y} \rangle} \mathbf{x} - \mathbf{y}, \mathbf{y} \rangle \quad \overset{(*)}{=} \quad \tfrac{\langle \mathbf{y}, \mathbf{y} \rangle}{\langle \mathbf{x}, \mathbf{y} \rangle} \langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{y}, \mathbf{y} \rangle \quad = \quad 0,$$

where (*) follows from r.2 and r.3 if V is over $\mathbb{R}$, or from c.2 and c.3 if V is over $\mathbb{C}$.

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Let $\langle\cdot,\cdot\rangle$ be a scalar product in a vector space V over $\mathbb{R}$ or $\mathbb{C}$, and let $||\cdot||$ be the norm in V induced by $\langle\cdot,\cdot\rangle$. Then

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for all $\mathbf{x},\mathbf{y}\in V$.

Proof. Fix $\mathbf{x},\mathbf{y}\in V$. WMA $\langle\mathbf{x},\mathbf{y}\rangle\neq 0$, for otherwise, the result is immediate. This implies that $\mathbf{x},\mathbf{y}\neq\mathbf{0}$, and consequently, $||\mathbf{x}||,||\mathbf{y}||\neq 0$. We set

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where (*) follows from r.2 and r.3 if V is over $\mathbb{R}$, or from c.2 and c.3 if V is over $\mathbb{C}$. Thus, $\mathbf{z}\perp\mathbf{y}$, and so by the Pythagorean theorem, $||\mathbf{z}+\mathbf{y}||^2=||\mathbf{z}||^2+||\mathbf{y}||^2$.

Proof (continued). Reminder: $\mathbf{z} = \frac{\langle \mathbf{y},\mathbf{y}\rangle}{\langle \mathbf{x},\mathbf{y}\rangle}\mathbf{x} - \mathbf{y}$; $||\mathbf{z} + \mathbf{y}||^2 = ||\mathbf{z}||^2 + ||\mathbf{y}||^2$. WTS $|\langle \mathbf{x}, \mathbf{y}\rangle| \leq ||\mathbf{x}||\ ||\mathbf{y}||$.

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\Longrightarrow \qquad ||\mathbf{z}+\mathbf{y}|| &= ||\tfrac{\langle \mathbf{y},\mathbf{y}\rangle}{\langle \mathbf{x},\mathbf{y}\rangle}\mathbf{x}|| \\
&= |\tfrac{\langle \mathbf{y},\mathbf{y}\rangle}{\langle \mathbf{x},\mathbf{y}\rangle}|\;||\mathbf{x}|| \\
&= \tfrac{|\langle \mathbf{y},\mathbf{y}\rangle|}{|\langle \mathbf{x},\mathbf{y}\rangle|}\;||\mathbf{x}|| \\
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&= \tfrac{||\mathbf{y}||^2}{|\langle \mathbf{x},\mathbf{y}\rangle|}\ ||\mathbf{x}||
\end{aligned}$$

$$\Longrightarrow \qquad \tfrac{||\mathbf{y}||^4}{|\langle \mathbf{x},\mathbf{y}\rangle|^2}\ ||\mathbf{x}||^2 = ||\mathbf{z}+\mathbf{y}||^2 = ||\mathbf{z}||^2 + ||\mathbf{y}||^2 \geq ||\mathbf{y}||^2$$

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Proof (continued). Reminder: $\mathbf{z} = \frac{\langle \mathbf{y},\mathbf{y}\rangle}{\langle \mathbf{x},\mathbf{y}\rangle}\mathbf{x} - \mathbf{y}$;
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Q.E.D.

The Cauchy–Schwarz inequality

Let $\langle\cdot,\cdot\rangle$ be a scalar product in a vector space V over $\mathbb{R}$ or $\mathbb{C}$, and let $||\cdot||$ be the norm in V induced by $\langle\cdot,\cdot\rangle$. Then

$$|\langle\mathbf{x},\mathbf{y}\rangle| \quad \leq \quad ||\mathbf{x}||\,||\mathbf{y}||$$

for all $\mathbf{x},\mathbf{y}\in V$.

- Cauchy–Schwarz: $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq ||\mathbf{x}||\ ||\mathbf{y}||$

Corollary 2.2

For all $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$, we have that

$$\Big(\sum_{i=1}^{n} x_i y_i \Big)^2 \quad \leq \quad \Big(\sum_{i=1}^{n} x_i^2 \Big) \Big(\sum_{i=1}^{n} y_i^2 \Big).$$

Proof.

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Proof. If we consider the standard scalar product in $\mathbb{R}^n$, the Cauchy-Schwarz inequality yields

$$|\sum_{i=1}^{n} x_i y_i| \;\leq\; \sqrt{\sum_{i=1}^{n} x_i^2} \sqrt{\sum_{i=1}^{n} y_i^2}.$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$. By squaring both sides, we obtain the desired inequality. Q.E.D.

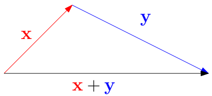
- Cauchy–Schwarz: $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq ||\mathbf{x}||\; ||\mathbf{y}||$

The triangle inequality

Let $\langle \cdot, \cdot \rangle$ be a scalar product in a vector space V over $\mathbb{R}$ or $\mathbb{C}$, and let $|| \cdot ||$ be the norm in V induced by $\langle \cdot, \cdot \rangle$. Then

$$||\mathbf{x} + \mathbf{y}|| \quad \leq \quad ||\mathbf{x}|| + ||\mathbf{y}||$$

for all $\mathbf{x}, \mathbf{y} \in V$.



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Proof. We prove the result for the case when V is a vector space over $\mathbb{C}$. The real case is similar, but easier.

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Proof. We prove the result for the case when V is a vector space over $\mathbb{C}$. The real case is similar, but easier.

We first remark that for all complex numbers $z = a + ib$ (where $a, b \in \mathbb{R}$), we have that

- $z + \overline{z} = 2a = 2\text{Re}(z)$;
- $\text{Re}(z) = a \leq |a| \leq \sqrt{a^2 + b^2} = |z|$.

Proof of the triangle inequality (continued). For $\mathbf{x}, \mathbf{y} \in V$:

$$\begin{aligned}
||\mathbf{x}+\mathbf{y}||^2 &= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle \\
&= \underbrace{\langle \mathbf{x}, \mathbf{x} \rangle}_{=||\mathbf{x}||^2} + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \underbrace{\langle \mathbf{y}, \mathbf{y} \rangle}_{=||\mathbf{y}||^2} && \text{by c.2 and c.2'} \\
&= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle \\
&= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + \langle \mathbf{x}, \mathbf{y} \rangle + \overline{\langle \mathbf{x}, \mathbf{y} \rangle} && \text{by c.4} \\
&= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2\text{Re}\Big(\langle \mathbf{x}, \mathbf{y} \rangle \Big) \\
&\leq ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2|\langle \mathbf{x}, \mathbf{y} \rangle| \\
&\leq ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2||\mathbf{x}||\;||\mathbf{y}|| && \text{by C-S}\leq \\
&= (||\mathbf{x}|| + ||\mathbf{y}||)^2.
\end{aligned}$$

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&\leq ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2||\mathbf{x}||\,||\mathbf{y}|| && \text{by C-S}\leq \\
&= (||\mathbf{x}|| + ||\mathbf{y}||)^2.
\end{aligned}$$

By taking the square root, we obtain $||\mathbf{x}+\mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$. Q.E.D.

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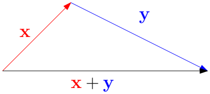

- Let us now define norms in general!

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Definition

A *norm* in a vector space V over $\mathbb{R}$ or $\mathbb{C}$ is a function $||\cdot|| : V \to \mathbb{R}$ that satisfies the following axioms:

n.1. for all vectors $\mathbf{x} \in V$, we have that $||\mathbf{x}|| \geq 0$, and equality holds if and only if $\mathbf{x} = \mathbf{0}$;

n.2. for all vectors $\mathbf{x} \in V$ and scalars α,[a] we have that $||\alpha\mathbf{x}|| = |\alpha| \; ||\mathbf{x}||$;

n.3. for all vectors $\mathbf{x}, \mathbf{y} \in V$, we have that $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$.

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[a] So, $\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{C}$, depending on whether V is a vector space over $\mathbb{R}$ or $\mathbb{C}$.

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- Any norm induced by a scalar product is indeed a norm, i.e. it satisfies n.1, n.2, and n.3.
  - n.1 follows from the construction, n.2 follows from Proposition 2.1, and n.3 follows from the triangle inequality.

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- Any norm induced by a scalar product is indeed a norm, i.e. it satisfies n.1, n.2, and n.3.
  - n.1 follows from the construction, n.2 follows from Proposition 2.1, and n.3 follows from the triangle inequality.
- In fact, n.3 itself is referred as the “triangle inequality” (but not it’s an axiom rather than a theorem).

- Other examples of norms

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- For a positive integer p, we define the *p-norm*, denoted by $||\cdot||_p$, on $\mathbb{R}^n$ by setting

  $$||\mathbf{x}||_p \quad := \quad \Big(\sum_{i=1}^{n} |x_i|^p \Big)^{\frac{1}{p}}$$

  for all $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ in $\mathbb{R}^n$.

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- We omit the proof of the fact that $||\cdot||_p$ is indeed a norm.

- Reminder: $||\mathbf{x}||_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \quad \forall \mathbf{x} = \left[\begin{array}{ccc} x_1 & \dots & x_n \end{array} \right]^T \in \mathbb{R}^n$.

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  $|| \cdot ||_1$ norm is sometimes called the "Manhattan norm." This is because streets and avenues in Manhattan form a perfect grid (more or less), and so $|| \cdot ||_1$ gives the actual walking distance between two places in Manhattan.

- Another norm of interest is the so called "Chebyshev distance" on $\mathbb{R}^n$, denoted by $||\cdot||_\infty$. It is defined by

  $$||\mathbf{x}||_\infty \quad := \quad \max\{|x_1|, \ldots, |x_n|\}$$

  for all vectors $\mathbf{x} = \left[\begin{array}{ccc} x_1 & \ldots & x_n \end{array}\right]^T$ in $\mathbb{R}^n$.

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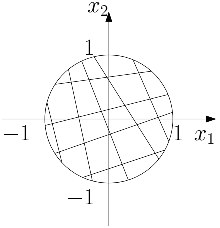
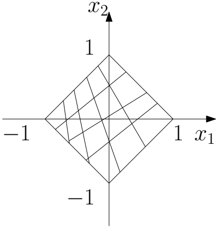


unit disk in $\mathbb{R}^2$
w.r.t. $||\cdot||_2$



unit disk in $\mathbb{R}^2$
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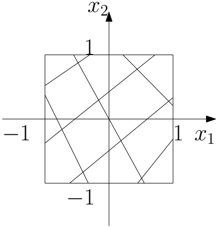

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- Finally, if you have studied calculus, recall that for $a, b \in \mathbb{R}$ such that $a < b$, $\mathcal{C}_{[a,b]}$ is the vector space (over $\mathbb{R}$) of all continuous functions from $[a, b]$ to $\mathbb{R}$.

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- For a real number $p \geq 1$, we have the norm $|| \cdot ||_p$ on $\mathcal{C}_{[a,b]}$ given by

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- Once again, we omit the proof of the fact that $|| \cdot ||_p$ (for a real number $p \geq 1$) and $|| \cdot ||_\infty$ really are norms on $\mathcal{C}_{[a,b]}$.