Linear Algebra 2

Lecture #10

Matrices of linear transformations. Change of basis (transition) matrices

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1. A review of coordinate vectors and standard matrices

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2. Matrices of linear transformations

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1. A review of coordinate vectors and standard matrices
2. Matrices of linear transformations
3. An example with polynomials (on the board)
4. A characterization of change of basis matrices. Similar matrices

(1) A review of coordinate vectors and standard matrices

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- For a field $\mathbb{F}$, a positive integer n, and an index $i \in \{1, \ldots, n\}$, the *i-th standard basis vector* in $\mathbb{F}^n$, denoted by $\mathbf{e}_i^n$, is the vector in $\mathbb{F}^n$ whose i-th entry is 1, and all of whose other entries are 0.

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  - When n is clear from context, we drop the superscript n and write $\mathbf{e}_i$ instead of $\mathbf{e}_i^n$.
- $\mathcal{E}_n = \{\mathbf{e}_1^n, \dots, \mathbf{e}_n^n\}$ is called the *standard basis* of $\mathbb{F}^n$.

Theorem 1.2 from Lecture Notes 4

Let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear transformation. Then there exists a unique matrix A (called the *standard matrix of f*) such that for all $\mathbf{u} \in \mathbb{F}^m$, we have that $f(\mathbf{u}) = A\mathbf{u}$. Moreover, the standard matrix A of f is given by

$$A = \left[\begin{array}{ccc} f(\mathbf{e}_1) & \dots & f(\mathbf{e}_m) \end{array} \right],$$

where $\mathbf{e}_1, \dots, \mathbf{e}_m$ are the standard basis vectors of $\mathbb{F}^m$.

Definition

For a positive integer n and a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of a vector space V over a field $\mathbb{F}$, the *coordinate vector* of a vector $\mathbf{v} \in V$ with respect to the basis $\mathcal{B}$ is the (unique) vector

$$[\mathbf{v}]_{\mathcal{B}} := \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

in $\mathbb{F}^n$ such that $\mathbf{v} = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n$.

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- Note that if $\mathbb{F}$ is a field, then $\forall \mathbf{u} \in \mathbb{F}^n$: $[\mathbf{u}]_{\mathcal{E}_n} = \mathbf{u}$.

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- Note that if $\mathbb{F}$ is a field, then $\forall \mathbf{u} \in \mathbb{F}^n$: $[\mathbf{u}]_{\mathcal{E}_n} = \mathbf{u}$.
- Indeed, if $\mathbf{u} = \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix}^T$, then $\mathbf{u} = u_1 \mathbf{e}_1 + \dots + u_n \mathbf{e}_n$, and consequently, $[\mathbf{u}]_{\mathcal{E}_n} = \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix}^T = \mathbf{u}$.

Theorem 2.2 from Lecture Notes 9

Let U and V be vector spaces over a field $\mathbb{F}$, and assume that U is non-trivial (i.e. contains at least one non-zero vector) and finite-dimensional. Let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis of U, and let $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$.[a] Then there exists a unique linear transformation $f : U \to V$ such that $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$. Moreover, this unique linear transformation $f : U \to V$ is defined as follows: for all $\mathbf{u} \in U$, we set

$$f(\mathbf{u}) \;=\; \alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n,$$

where $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ are such that $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 & \dots & \alpha_n \end{bmatrix}^T$.

[a] Here, $\mathbf{v}_1, \dots, \mathbf{v}_n$ are arbitrary vectors in V. They are not necessarily pairwise distinct.

(2) Matrices of linear transformations

Theorem 2.1

Let U and V be non-trivial, finite-dimensional vector spaces over a field $\mathbb{F}$. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ be a basis for U, let $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be a basis for V, and let $f : U \to V$ be a linear transformation. Then exists a unique matrix in $\mathbb{F}^{n \times m}$, denoted by ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ and called the *matrix of f with respect to $\mathcal{B}$ and $\mathcal{C}$*, such that for all $\mathbf{u} \in U$, we have that

$$ {}_{\mathcal{C}}[f]_{\mathcal{B}} \, [\mathbf{u}]_{\mathcal{B}} \;=\; [f(\mathbf{u})]_{\mathcal{C}}. $$

Moreover, the matrix ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is given by

$$ {}_{\mathcal{C}}[f]_{\mathcal{B}} \;=\; \begin{bmatrix} [f(\mathbf{b}_1)]_{\mathcal{C}} & \dots & [f(\mathbf{b}_m)]_{\mathcal{C}} \end{bmatrix}. $$

- Note that if $\mathbb{F}$ is a field and $f : \mathbb{F}^m \to \mathbb{F}^n$ is a linear function, then ${}_{\mathcal{E}_n}[f]_{\mathcal{E}_m}$ is precisely the standard matrix of f.

Proof of Theorem 2.1.

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Fix $\mathbf{u} \in U$. WTS $\begin{bmatrix} [f(\mathbf{b}_1)]_{\mathcal{C}} & \dots & [f(\mathbf{b}_m)]_{\mathcal{C}} \end{bmatrix} [\mathbf{u}]_{\mathcal{B}} = [f(\mathbf{u})]_{\mathcal{C}}$.

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Set $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} \beta_1 & \dots & \beta_m \end{bmatrix}^T$, so that $\mathbf{u} = \beta_1 \mathbf{b}_1 + \dots + \beta_m \mathbf{b}_m$.

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We now have the following:

$$\begin{aligned}
& \begin{bmatrix} [f(\mathbf{b}_1)]_{\mathcal{C}} & \dots & [f(\mathbf{b}_m)]_{\mathcal{C}} \end{bmatrix} [\mathbf{u}]_{\mathcal{B}} \\
= \quad & \begin{bmatrix} [f(\mathbf{b}_1)]_{\mathcal{C}} & \dots & [f(\mathbf{b}_m)]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} \\
= \quad & \beta_1 [f(\mathbf{b}_1)]_{\mathcal{C}} + \dots + \beta_m [f(\mathbf{b}_m)]_{\mathcal{C}} \\
= \quad & [\beta_1 f(\mathbf{b}_1) + \dots + \beta_m f(\mathbf{b}_m)]_{\mathcal{C}} \\
= \quad & [f(\beta_1 \mathbf{b}_1 + \dots + \beta_m \mathbf{b}_m)]_{\mathcal{C}} \\
= \quad & [f(\mathbf{u})]_{\mathcal{C}}.
\end{aligned}$$

Proof of Theorem 2.1 (continued). We now prove uniqueness.

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Fix any matrix $A \in \mathbb{F}^{n \times m}$ s.t. $\forall \mathbf{u} \in U$: $A[\mathbf{u}]_{\mathcal{B}} = [f(\mathbf{u})]_{\mathcal{C}}$.

Proof of Theorem 2.1 (continued). We now prove uniqueness.

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WTS $A = \left[\begin{array}{ccc} [f(\mathbf{b}_1)]_{\mathcal{C}} & \dots & [f(\mathbf{b}_m)]_{\mathcal{C}} \end{array}\right]$.

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Fix $i \in \{1, \dots, m\}$. WTS $\mathbf{a}_i = [f(\mathbf{b}_i)]_{\mathcal{C}}$

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Note that $[\mathbf{b}_i]_{\mathcal{B}} = \mathbf{e}_i^m$.

Proof of Theorem 2.1 (continued). We now prove uniqueness.

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Note that $[\mathbf{b}_i]_{\mathcal{B}} = \mathbf{e}_i^m$. But now

$$\mathbf{a}_i \;=\; A\mathbf{e}_i^m \;=\; A[\mathbf{b}_i]_{\mathcal{B}} \;=\; [f(\mathbf{b}_i)]_{\mathcal{C}}.$$

We have now shown that $A = \left[\begin{array}{ccc} [f(\mathbf{b}_1)]_{\mathcal{C}} & \dots & [f(\mathbf{b}_m)]_{\mathcal{C}} \end{array} \right]$. This proves uniqueness. Q.E.D.

Theorem 2.1

Let U and V be non-trivial, finite-dimensional vector spaces over a field $\mathbb{F}$. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ be a basis for U, let $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be a basis for V, and let $f : U \to V$ be a linear transformation. Then exists a unique matrix in $\mathbb{F}^{n \times m}$, denoted by ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ and called the *matrix of f with respect to $\mathcal{B}$ and $\mathcal{C}$*, such that for all $\mathbf{u} \in U$, we have that

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Moreover, the matrix ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is given by

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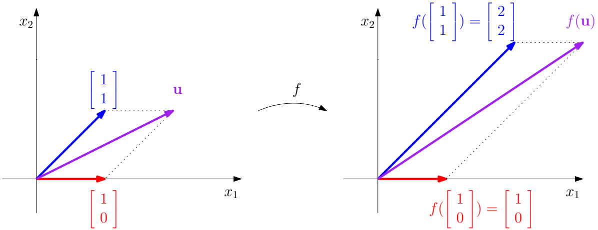
- Note that if $\mathbb{F}$ is a field and $f : \mathbb{F}^m \to \mathbb{F}^n$ is a linear function, then ${}_{\mathcal{E}_n}[f]_{\mathcal{E}_m}$ is precisely the standard matrix of f.

Example 2.2

Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$ and the unique linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Find the matrix ${}_{\mathcal{B}}[f]_{\mathcal{B}}$.



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Solution. Using the formula from Theorem 2.1, we get that

$$\begin{aligned} {}_{\mathcal{B}}[f]_{\mathcal{B}} &= \begin{bmatrix} [f(\begin{bmatrix} 1 \\ 0 \end{bmatrix})]_{\mathcal{B}} & [f(\begin{bmatrix} 1 \\ 1 \end{bmatrix})]_{\mathcal{B}} \end{bmatrix} \\ &= \begin{bmatrix} [\begin{bmatrix} 1 \\ 0 \end{bmatrix}]_{\mathcal{B}} & [\begin{bmatrix} 2 \\ 2 \end{bmatrix}]_{\mathcal{B}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}. \end{aligned}$$

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Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$ and the unique linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Find the matrix ${}_{\mathcal{B}}[f]_{\mathcal{B}}$.

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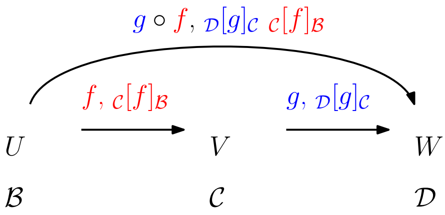
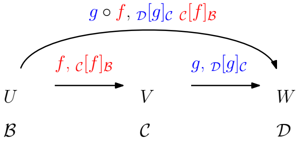
- What if we want to compute the standard matrix of the linear transformation f from Example 2.2?
- This can be done, but we first need some theory!

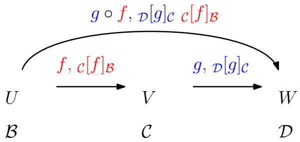
Proposition 2.3

Let U, V, and W be non-trivial, finite-dimensional vector spaces over a field $\mathbb{F}$. Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ be a basis for U, let $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be a basis for V, and let $\mathcal{D} = \{\mathbf{d}_1, \ldots, \mathbf{d}_p\}$ be a basis for W. Let $f : U \to V$ and $g : V \to W$ be linear transformations. ${}_{\mathcal{D}}[g \circ f]_{\mathcal{B}} = {}_{\mathcal{D}}[g]_{\mathcal{C}}\ {}_{\mathcal{C}}[f]_{\mathcal{B}}$.





Proof of Proposition 2.3. WTS ${}_{\mathcal{D}}[g \circ f]_{\mathcal{B}} = {}_{\mathcal{D}}[g]_{\mathcal{C}}\ {}_{\mathcal{C}}[f]_{\mathcal{B}}$.



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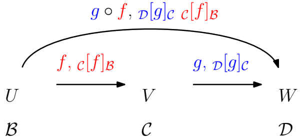
By Theorem 2.1, ${}_{\mathcal{D}}[g \circ f]_{\mathcal{B}}$ is the unique matrix in $\mathbb{F}^{p \times m}$ s.t. $\forall \mathbf{u} \in U$: ${}_{\mathcal{D}}[g \circ f]_{\mathcal{B}}\ [\mathbf{u}]_{\mathcal{B}} = [(g \circ f)(\mathbf{u})]_{\mathcal{D}}$.

So, it suffices to show that $\forall \mathbf{u} \in U$:
$\left({}_{\mathcal{D}}[g]_{\mathcal{C}}\ {}_{\mathcal{C}}[f]_{\mathcal{B}}\right)[\mathbf{u}]_{\mathcal{B}} = [(g \circ f)(\mathbf{u})]_{\mathcal{D}}$.



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So, it suffices to show that $\forall \mathbf{u} \in U$:
$\Big({}_{\mathcal{D}}[g]_{\mathcal{C}}\ {}_{\mathcal{C}}[f]_{\mathcal{B}}\Big)[\mathbf{u}]_{\mathcal{B}} = [(g \circ f)(\mathbf{u})]_{\mathcal{D}}$.

And indeed, $\forall \mathbf{u} \in U$, we have that

$$\begin{aligned}
\Big({}_{\mathcal{D}}[g]_{\mathcal{C}}\ {}_{\mathcal{C}}[f]_{\mathcal{B}}\Big)[\mathbf{u}]_{\mathcal{B}} &= {}_{\mathcal{D}}[g]_{\mathcal{C}}\Big({}_{\mathcal{C}}[f]_{\mathcal{B}}\ [\mathbf{u}]_{\mathcal{B}}\Big) \\
&= {}_{\mathcal{D}}[g]_{\mathcal{C}}\ [f(\mathbf{u})]_{\mathcal{C}} \\
&= [g(f(\mathbf{u}))]_{\mathcal{D}} \\
&= [(g \circ f)(\mathbf{u})]_{\mathcal{D}}.
\end{aligned}$$

Q.E.D.

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Proposition 2.4

Let U and V be non-trivial, finite-dimensional vector spaces over a field $\mathbb{F}$. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ be a basis for U, let $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be a basis for V, and let $f : U \to V$ be a linear transformation. Then all the following hold:

(a) $\text{rank}(f) = \text{rank}\Big({}_{\mathcal{C}}[f]_{\mathcal{B}}\Big)$;[a]

(b) f is an isomorphism if and only if ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is invertible (and in particular, square);

(c) if f is an isomorphism, then ${}_{\mathcal{B}}[f^{-1}]_{\mathcal{C}} = \Big({}_{\mathcal{C}}[f]_{\mathcal{B}}\Big)^{-1}$.

[a] Recall that, by definition, $\text{rank}(f) = \dim\Big(\text{Im}(f)\Big)$.

Proof of Proposition 2.4(a). WTS $\text{rank}(f) = \text{rank}\Big({}_{\mathcal{C}}[f]_{\mathcal{B}}\Big)$.

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Proof of Proposition 2.4(a). WTS $\text{rank}(f) = \text{rank}\Big({}_{\mathcal{C}}[f]_{\mathcal{B}}\Big)$.

By Theorem 2.1, ${}_{\mathcal{C}}[f]_{\mathcal{B}} = \Big[\ [f(\mathbf{b}_1)]_{\mathcal{C}} \quad \dots \quad [f(\mathbf{b}_m)]_{\mathcal{C}}\ \Big]$.

We now compute:

$$\begin{array}{rcl} \text{rank}(f) & = & \dim(\text{Im}(f)) \\ & = & \dim\Big(\text{Span}\big(f(\mathbf{b}_1), \dots, f(\mathbf{b}_m)\big)\Big) \\ & = & \dim\Big(\text{Span}\big([f(\mathbf{b}_1)]_{\mathcal{C}}, \dots, [f(\mathbf{b}_m)]_{\mathcal{C}}\big)\Big) \\ & = & \dim\Big(\text{Col}\Big(\Big[\ [f(\mathbf{b}_1)]_{\mathcal{C}} \quad \dots \quad [f(\mathbf{b}_m)]_{\mathcal{C}}\ \Big]\Big)\Big) \\ & = & \dim\Big(\text{Col}\big({}_{\mathcal{C}}[f]_{\mathcal{B}}\big)\Big) \\ & = & \text{rank}\big({}_{\mathcal{C}}[f]_{\mathcal{B}}\big) \end{array}$$

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By (a), it follows that $\text{rank}({}_{\mathcal{C}}[f]_{\mathcal{B}}) = n$. Since ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is an $n \times n$ matrix of rank n, we know that ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is invertible.

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Now $f : U \to V$ is an onto linear transformation between vector spaces of the same finite dimension;

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Now $f : U \to V$ is an onto linear transformation between vector spaces of the same finite dimension; by Corollary 1.2 from Lecture Notes 9, it follows that f is also one-to-one.

So, f is an isomorphism.

Proof of Proposition 2.4(c). Suppose that f is an isomorphism. Then by (b), ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is invertible. WTS ${}_{\mathcal{B}}[f^{-1}]_{\mathcal{C}} = \left({}_{\mathcal{C}}[f]_{\mathcal{B}}\right)^{-1}$.

Proof of Proposition 2.4(c). Suppose that f is an isomorphism. Then by (b), ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is invertible. WTS ${}_{\mathcal{B}}[f^{-1}]_{\mathcal{C}} = \left({}_{\mathcal{C}}[f]_{\mathcal{B}}\right)^{-1}$.

It now suffices to show that for all $\mathbf{v} \in V$, we have that $\left({}_{\mathcal{C}}[f]_{\mathcal{B}}\right)^{-1}[\mathbf{v}]_{\mathcal{C}} = [f^{-1}(\mathbf{v})]_{\mathcal{B}}$, for then Theorem 2.1 will imply that ${}_{\mathcal{B}}[f^{-1}]_{\mathcal{C}} = \left({}_{\mathcal{C}}[f]_{\mathcal{B}}\right)^{-1}$.

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Then ${}_{\mathcal{C}}[f]_{\mathcal{B}}\ [\mathbf{u}]_{\mathcal{B}} = [f(\mathbf{u})]_{\mathcal{C}}$.

Proof of Proposition 2.4(c). Suppose that f is an isomorphism. Then by (b), ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is invertible. WTS ${}_{\mathcal{B}}[f^{-1}]_{\mathcal{C}} = \left({}_{\mathcal{C}}[f]_{\mathcal{B}}\right)^{-1}$.

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Then ${}_{\mathcal{C}}[f]_{\mathcal{B}}\ [\mathbf{u}]_{\mathcal{B}} = [f(\mathbf{u})]_{\mathcal{C}}$.

Since ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is invertible, we can multiply both sides by $\left({}_{\mathcal{C}}[f]_{\mathcal{B}}\right)^{-1}$.

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We obtain $[\mathbf{u}]_{\mathcal{B}} = \left({}_{\mathcal{C}}[f]_{\mathcal{B}}\right)^{-1}[f(\mathbf{u})]_{\mathcal{C}}$, i.e. $[f^{-1}(\mathbf{v})]_{\mathcal{B}} = \left({}_{\mathcal{C}}[f]_{\mathcal{B}}\right)^{-1}[\mathbf{v}]_{\mathcal{C}}$. Q.E.D.

Proposition 2.4

Let U and V be non-trivial, finite-dimensional vector spaces over a field $\mathbb{F}$. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ be a basis for U, let $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be a basis for V, and let $f : U \to V$ be a linear transformation. Then all the following hold:

a. $\text{rank}(f) = \text{rank}\Big({}_{\mathcal{C}}[f]_{\mathcal{B}}\Big)$;[a]
b. f is an isomorphism if and only if ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is invertible (and in particular, square);
c. if f is an isomorphism, then ${}_{\mathcal{B}}[f^{-1}]_{\mathcal{C}} = \Big({}_{\mathcal{C}}[f]_{\mathcal{B}}\Big)^{-1}$.

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[a] Recall that, by definition, $\text{rank}(f) = \dim\Big(\text{Im}(f)\Big)$.

Definition

Given a non-trivial, finite-dimensional vector space V over a field $\mathbb{F}$, and bases $\mathcal{B}$ and $\mathcal{C}$ of V, we call the matrix ${}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$ the *change of basis matrix from* $\mathcal{B}$ *to* $\mathcal{C}$ or the *transition matrix from* $\mathcal{B}$ *to* $\mathcal{C}$.

- Note: ${}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}\ [\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{C}} \quad \forall \mathbf{v} \in V$.

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Proposition 2.5

Let V be a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be bases of V. Then the change of basis matrices ${}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$ and ${}_{\mathcal{B}}[\mathrm{Id}_V]_{\mathcal{C}}$ are invertible, and moreover, they are each other's inverses.

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Proof. Clearly, Id_V is an isomorphism, and moreover, $\mathrm{Id}_V^{-1} = \mathrm{Id}_V$. So, the result follows immediately from Proposition 2.4.

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Theorem 2.7

Let $\mathbb{F}$ be a field, and let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be two bases for $\mathbb{F}^n$. Set $B := \begin{bmatrix} \mathbf{b}_1 & \ldots & \mathbf{b}_n \end{bmatrix}$ and $C := \begin{bmatrix} \mathbf{c}_1 & \ldots & \mathbf{c}_n \end{bmatrix}$. Then ${}_{\mathcal{C}}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{B}} = C^{-1}B$.

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- Let's first prove a lemma!

Lemma 2.6

Let $\mathbb{F}$ be a field, let $\mathcal{E}_n = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{F}^n$, and let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be any basis for $\mathbb{F}^n$. Set $B := \left[\begin{array}{ccc} \mathbf{b}_1 & \dots & \mathbf{b}_n \end{array} \right]$. Then B is invertible, and moreover, ${}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{B}} = B$ and ${}_{\mathcal{B}}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{E}_n} = B^{-1}$.

Proof.

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Proof. WTS ${}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{B}} = B$.

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Proof. WTS ${}_{\mathcal{E}_n}[\text{Id}_{\mathbb{F}^n}]_{\mathcal{B}} = B$. In view of Theorem 2.1, it suffices to show that for all $\mathbf{v} \in \mathbb{F}^n$, we have that $B[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{E}_n}$.

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So, fix $\mathbf{v} \in \mathbb{F}^n$, and set $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} \beta_1 & \dots & \beta_n \end{bmatrix}^T$; then $\mathbf{v} = \beta_1\mathbf{b}_1 + \dots + \beta_n\mathbf{b}_n$.

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$$B[\mathbf{v}]_{\mathcal{B}} \;=\; \begin{bmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \;=\; \mathbf{v} \;=\; [\mathbf{v}]_{\mathcal{E}_n}.$$

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Let $\mathbb{F}$ be a field, let $\mathcal{E}_n = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{F}^n$, and let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be any basis for $\mathbb{F}^n$. Set $B := \left[\begin{array}{ccc} \mathbf{b}_1 & \dots & \mathbf{b}_n \end{array} \right]$. Then B is invertible, and moreover, ${}_{\mathcal{E}_n}[\text{Id}_{\mathbb{F}^n}]_{\mathcal{B}} = B$ and ${}_{\mathcal{B}}[\text{Id}_{\mathbb{F}^n}]_{\mathcal{E}_n} = B^{-1}$.

Proof. WTS ${}_{\mathcal{E}_n}[\text{Id}_{\mathbb{F}^n}]_{\mathcal{B}} = B$. In view of Theorem 2.1, it suffices to show that for all $\mathbf{v} \in \mathbb{F}^n$, we have that $B[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{E}_n}$.

So, fix $\mathbf{v} \in \mathbb{F}^n$, and set $[\mathbf{v}]_{\mathcal{B}} = \left[\begin{array}{ccc} \beta_1 & \dots & \beta_n \end{array} \right]^T$; then $\mathbf{v} = \beta_1 \mathbf{b}_1 + \dots + \beta_n \mathbf{b}_n$. Also, $[\mathbf{v}]_{\mathcal{E}_n} = \mathbf{v}$. Then

$$B[\mathbf{v}]_{\mathcal{B}} \quad = \quad \left[\begin{array}{ccc} \mathbf{b}_1 & \dots & \mathbf{b}_n \end{array} \right] \left[\begin{array}{c} \beta_1 \\ \vdots \\ \beta_n \end{array} \right] \quad = \quad \mathbf{v} \quad = \quad [\mathbf{v}]_{\mathcal{E}_n}.$$

This proves that ${}_{\mathcal{E}_n}[\text{Id}_{\mathbb{F}^n}]_{\mathcal{B}} = B$.

Lemma 2.6

Let $\mathbb{F}$ be a field, let $\mathcal{E}_n = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{F}^n$, and let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be any basis for $\mathbb{F}^n$. Set $B := \begin{bmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_n \end{bmatrix}$. Then B is invertible, and moreover, ${}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{B}} = B$ and ${}_{\mathcal{B}}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{E}_n} = B^{-1}$.

Proof. WTS ${}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{B}} = B$. In view of Theorem 2.1, it suffices to show that for all $\mathbf{v} \in \mathbb{F}^n$, we have that $B[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{E}_n}$.

So, fix $\mathbf{v} \in \mathbb{F}^n$, and set $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} \beta_1 & \dots & \beta_n \end{bmatrix}^T$; then $\mathbf{v} = \beta_1 \mathbf{b}_1 + \dots + \beta_n \mathbf{b}_n$. Also, $[\mathbf{v}]_{\mathcal{E}_n} = \mathbf{v}$. Then

$$B[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} = \mathbf{v} = [\mathbf{v}]_{\mathcal{E}_n}.$$

This proves that ${}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{B}} = B$. The fact that B is invertible and that ${}_{\mathcal{B}}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{E}_n} = B^{-1}$ now follows from Proposition 2.5.

Lemma 2.6

Let $\mathbb{F}$ be a field, let $\mathcal{E}_n = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{F}^n$, and let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be any basis for $\mathbb{F}^n$. Set $B := \begin{bmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_n \end{bmatrix}$. Then B is invertible, and moreover, ${}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{B}} = B$ and ${}_{\mathcal{B}}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{E}_n} = B^{-1}$.

Lemma 2.6

Let $\mathbb{F}$ be a field, let $\mathcal{E}_n = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{F}^n$, and let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be any basis for $\mathbb{F}^n$. Set $B := \begin{bmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_n \end{bmatrix}$. Then B is invertible, and moreover, ${}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{B}} = B$ and ${}_{\mathcal{B}}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{E}_n} = B^{-1}$.

Theorem 2.7

Let $\mathbb{F}$ be a field, and let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be two bases for $\mathbb{F}^n$. Set $B := \begin{bmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_n \end{bmatrix}$ and $C := \begin{bmatrix} \mathbf{c}_1 & \dots & \mathbf{c}_n \end{bmatrix}$. Then ${}_{\mathcal{C}}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{B}} = C^{-1}B$.

Proof.

Lemma 2.6

Let $\mathbb{F}$ be a field, let $\mathcal{E}_n = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{F}^n$, and let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be any basis for $\mathbb{F}^n$. Set $B := \begin{bmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_n \end{bmatrix}$. Then B is invertible, and moreover, ${}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{B}} = B$ and ${}_{\mathcal{B}}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{E}_n} = B^{-1}$.

Theorem 2.7

Let $\mathbb{F}$ be a field, and let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be two bases for $\mathbb{F}^n$. Set $B := \begin{bmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_n \end{bmatrix}$ and $C := \begin{bmatrix} \mathbf{c}_1 & \dots & \mathbf{c}_n \end{bmatrix}$. Then ${}_{\mathcal{C}}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{B}} = C^{-1}B$.

Proof. We observe that

$$\begin{aligned}
{}_{\mathcal{C}}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{B}} &= {}_{\mathcal{C}}[\mathrm{Id}_{\mathbb{F}^n} \circ \mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{B}} \\
&= {}_{\mathcal{C}}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{E}_n}\ {}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{B}} && \text{by Proposition 2.3} \\
&= C^{-1}B && \text{by Lemma 2.6}
\end{aligned}$$

Q.E.D.

- Reminder:

Example 2.2

Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$ and the unique linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Find the matrix ${}_{\mathcal{B}}[f]_{\mathcal{B}}$.

Solution. ${}_{\mathcal{B}}[f]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

- Reminder:

Example 2.2

Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$ and the unique linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Find the matrix ${}_{\mathcal{B}}[f]_{\mathcal{B}}$.

Solution. ${}_{\mathcal{B}}[f]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

- Let's now find the standard matrix for f!

Example 2.8

Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$ and the unique linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Find the standard matrix of f.

Solution.

Example 2.8

Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$ and the unique linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Find the standard matrix of f.

Solution.

Example 2.8

Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$ and the unique linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Find the standard matrix of f.

Solution. In Example 2.2, we saw that ${}_{\mathcal{B}}[f]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

Example 2.8

Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$ and the unique linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Find the standard matrix of f.

Solution. In Example 2.2, we saw that ${}_{\mathcal{B}}[f]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Now, set $B := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and note that $B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

Solution (continued). Then the standard matrix of f is

$$
\begin{aligned}
{}_{\mathcal{E}_2}[f]_{\mathcal{E}_2} &= {}_{\mathcal{E}_2}[\mathsf{Id}_{\mathbb{R}^2} \circ f \circ \mathsf{Id}_{\mathbb{R}^2}]_{\mathcal{E}_2} \\
&= {}_{\mathcal{E}_2}[\mathsf{Id}_{\mathbb{R}^2}]_{\mathcal{B}}\ {}_{\mathcal{B}}[f]_{\mathcal{B}}\ {}_{\mathcal{B}}[\mathsf{Id}_{\mathbb{R}^2}]_{\mathcal{E}_2} && \text{by Proposition 2.3} \\
&= B\ {}_{\mathcal{B}}[f]_{\mathcal{B}}\ B^{-1} && \text{by Lemma 2.6} \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.
\end{aligned}
$$

Example 2.8

Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$ and the unique linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Find the standard matrix of f.

Solution (continued). Reminder: ${}_{\mathcal{E}_2}[f]_{\mathcal{E}_2} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

Example 2.8

Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$ and the unique linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Find the standard matrix of f.

Solution (continued). Reminder: ${}_{\mathcal{E}_2}[f]_{\mathcal{E}_2} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

Optional. Let us check that our answer is correct!

- $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = f(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$;
- $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = f(\begin{bmatrix} 1 \\ 1 \end{bmatrix})$.

So, our answer is correct.

(3) An example with polynomials (on the board)

(3) An example with polynomials (on the board)

Example 3.1

As usual, let $\mathbb{P}^3_{\mathbb{Z}_2}$ be the vector space (over $\mathbb{Z}_2$) of all polynomials with coefficients in $\mathbb{Z}_2$ and of degree at most 3.

(a) Prove that there exists a unique linear transformation $f : \mathbb{P}^3_{\mathbb{Z}_2} \to \mathbb{P}_{\mathbb{Z}_2}$ that satisfies all the following:
   - $f(1) = x^4$;
   - $f(x+1) = x^3 + x^2$;
   - $f(x^2+x+1) = x^2 + 1$;
   - $f(x^3+x^2+x+1) = x$.

(b) Find rank(f), and determine whether f is one-to-one, where f is the linear transformation from part (a).

(c) Find the formula for the linear transformation $f : \mathbb{P}^3_{\mathbb{Z}_2} \to \mathbb{P}_{\mathbb{Z}_2}$ from part (a), that is, fill in the blank in the following:

$$f(a_3x^3 + a_2x^2 + a_1x + a_0) \quad = \quad \_\_\_\_ \qquad \forall a_0, a_1, a_2, a_3 \in \mathbb{Z}_2.$$

(4) **A characterization of change of basis matrices. Similar matrices**

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- As the following proposition shows, change of basis matrices are precisely the invertible matrices.

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- As the following proposition shows, change of basis matrices are precisely the invertible matrices.

Proposition 4.1

Let n be a positive integer, let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $A \in \mathbb{F}^{n \times n}$. Then the following are equivalent:

(i) there exist bases $\mathcal{B}$ and $\mathcal{C}$ for V such that $A = {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$;

(ii) A is invertible.

Proposition 4.1

Let n be a positive integer, let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $A \in \mathbb{F}^{n \times n}$. Then the following are equivalent:

1. there exist bases $\mathcal{B}$ and $\mathcal{C}$ for V such that $A = {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$;
2. A is invertible.

Proof.

Proposition 4.1

Let n be a positive integer, let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $A \in \mathbb{F}^{n \times n}$. Then the following are equivalent:

1. (i) there exist bases $\mathcal{B}$ and $\mathcal{C}$ for V such that $A = {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$;
2. (ii) A is invertible.

Proof. Since $\mathrm{Id}_{\mathbb{F}^n} : \mathbb{F}^n \to \mathbb{F}^n$ is an isomorphism, "(i) $\Longrightarrow$ (ii)" follows from Proposition 2.4.

Proposition 4.1

Let n be a positive integer, let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $A \in \mathbb{F}^{n\times n}$. Then the following are equivalent:

(i) there exist bases $\mathcal{B}$ and $\mathcal{C}$ for V such that $A = {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$;

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Proof. Since $\mathrm{Id}_{\mathbb{F}^n} : \mathbb{F}^n \to \mathbb{F}^n$ is an isomorphism, "(i) $\implies$ (ii)" follows from Proposition 2.4.

Suppose now that (ii) holds.

Proposition 4.1

Let n be a positive integer, let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $A \in \mathbb{F}^{n\times n}$. Then the following are equivalent:

1. (i) there exist bases $\mathcal{B}$ and $\mathcal{C}$ for V such that $A = {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$;
2. (ii) A is invertible.

Proof. Since $\mathrm{Id}_{\mathbb{F}^n} : \mathbb{F}^n \to \mathbb{F}^n$ is an isomorphism, "(i) $\Longrightarrow$ (ii)" follows from Proposition 2.4.

Suppose now that (ii) holds. Set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$.

Proposition 4.1

Let n be a positive integer, let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $A \in \mathbb{F}^{n\times n}$. Then the following are equivalent:

1. (i) there exist bases $\mathcal{B}$ and $\mathcal{C}$ for V such that $A = {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$;
2. (ii) A is invertible.

Proof. Since $\mathrm{Id}_{\mathbb{F}^n} : \mathbb{F}^n \to \mathbb{F}^n$ is an isomorphism, "(i) $\implies$ (ii)" follows from Proposition 2.4.

Suppose now that (ii) holds. Set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$. Since A is invertible, $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is a basis of $\mathbb{F}^n$. Then by Lemma 2.6, $A = {}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{A}}$.

Proposition 4.1

Let n be a positive integer, let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $A \in \mathbb{F}^{n \times n}$. Then the following are equivalent:

1. (i) there exist bases $\mathcal{B}$ and $\mathcal{C}$ for V such that $A = {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$;
2. (ii) A is invertible.

Proof. Since $\mathrm{Id}_{\mathbb{F}^n} : \mathbb{F}^n \to \mathbb{F}^n$ is an isomorphism, "(i) $\Longrightarrow$ (ii)" follows from Proposition 2.4.

Suppose now that (ii) holds. Set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$. Since A is invertible, $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is a basis of $\mathbb{F}^n$. Then by Lemma 2.6, $A = {}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{A}}$.

So far, we have shown that A is a change of basis matrix between two bases of $\mathbb{F}^n$.

Proposition 4.1

Let n be a positive integer, let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $A \in \mathbb{F}^{n\times n}$. Then the following are equivalent:

1. (i) there exist bases $\mathcal{B}$ and $\mathcal{C}$ for V such that $A = {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$;
2. (ii) A is invertible.

Proof. Since $\mathrm{Id}_{\mathbb{F}^n} : \mathbb{F}^n \to \mathbb{F}^n$ is an isomorphism, "(i) $\Longrightarrow$ (ii)" follows from Proposition 2.4.

Suppose now that (ii) holds. Set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$. Since A is invertible, $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is a basis of $\mathbb{F}^n$. Then by Lemma 2.6, $A = {}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{A}}$.

So far, we have shown that A is a change of basis matrix between two bases of $\mathbb{F}^n$. However, we need to show that A is a change of basis matrix between two bases of V.

Proposition 4.1

Let n be a positive integer, let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $A \in \mathbb{F}^{n \times n}$. Then the following are equivalent:

1. (i) there exist bases $\mathcal{B}$ and $\mathcal{C}$ for V such that $A = {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$;
2. (ii) A is invertible.

Proof. Since $\mathrm{Id}_{\mathbb{F}^n} : \mathbb{F}^n \to \mathbb{F}^n$ is an isomorphism, "(i) $\implies$ (ii)" follows from Proposition 2.4.

Suppose now that (ii) holds. Set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$. Since A is invertible, $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is a basis of $\mathbb{F}^n$. Then by Lemma 2.6, $A = {}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{A}}$.

So far, we have shown that A is a change of basis matrix between two bases of $\mathbb{F}^n$. However, we need to show that A is a change of basis matrix between two bases of V. For this, we use an arbitrary isomorphism $g : V \to \mathbb{F}^n$ (the isomorphism g exists because V and $\mathbb{F}^n$ are both n-dimensional vector spaces over $\mathbb{F}$).

Proposition 4.1

Let n be a positive integer, let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $A \in \mathbb{F}^{n \times n}$. Then the following are equivalent:

1. there exist bases $\mathcal{B}$ and $\mathcal{C}$ for V such that $A = {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$;
2. A is invertible.

Proof (continued). Reminder: (ii) holds; $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$; $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is a basis of $\mathbb{F}^n$; $A = {}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{A}}$; $g : V \to \mathbb{F}^n$ is an isomorphism.

Proposition 4.1

Let n be a positive integer, let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $A \in \mathbb{F}^{n\times n}$. Then the following are equivalent:

1. (i) there exist bases $\mathcal{B}$ and $\mathcal{C}$ for V such that $A = {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$;
2. (ii) A is invertible.

Proof (continued). Reminder: (ii) holds; $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$; $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is a basis of $\mathbb{F}^n$; $A = {}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{A}}$; $g : V \to \mathbb{F}^n$ is an isomorphism.

$\forall i \in \{1, \dots, n\}$, let $\mathbf{b}_i = g^{-1}(\mathbf{a}_i)$ and $\mathbf{c}_i = g^{-1}(\mathbf{e}_i)$.

Proposition 4.1

Let n be a positive integer, let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $A \in \mathbb{F}^{n \times n}$. Then the following are equivalent:

1. (i) there exist bases $\mathcal{B}$ and $\mathcal{C}$ for V such that $A = {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$;
2. (ii) A is invertible.

Proof (continued). Reminder: (ii) holds; $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$; $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is a basis of $\mathbb{F}^n$; $A = {}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{A}}$; $g : V \to \mathbb{F}^n$ is an isomorphism.
$\forall i \in \{1, \dots, n\}$, let $\mathbf{b}_i = g^{-1}(\mathbf{a}_i)$ and $\mathbf{c}_i = g^{-1}(\mathbf{e}_i)$. Then $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ are bases for V.

Proposition 4.1

Let n be a positive integer, let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $A \in \mathbb{F}^{n\times n}$. Then the following are equivalent:

1. (i) there exist bases $\mathcal{B}$ and $\mathcal{C}$ for V such that $A = {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$;
2. (ii) A is invertible.

Proof (continued). Reminder: (ii) holds; $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$; $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is a basis of $\mathbb{F}^n$; $A = {}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{A}}$; $g : V \to \mathbb{F}^n$ is an isomorphism.
$\forall i \in \{1, \dots, n\}$, let $\mathbf{b}_i = g^{-1}(\mathbf{a}_i)$ and $\mathbf{c}_i = g^{-1}(\mathbf{e}_i)$. Then $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ are bases for V. Note also that ${}_{\mathcal{A}}[g]_{\mathcal{B}} = I_n$ and ${}_{\mathcal{C}}[g^{-1}]_{\mathcal{E}_n} = I_n$.

Proposition 4.1

Let n be a positive integer, let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $A \in \mathbb{F}^{n\times n}$. Then the following are equivalent:

1. (i) there exist bases $\mathcal{B}$ and $\mathcal{C}$ for V such that $A = {}_{\mathcal{C}}[\text{Id}_V]_{\mathcal{B}}$;
2. (ii) A is invertible.

Proof (continued). Reminder: (ii) holds; $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$; $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is a basis of $\mathbb{F}^n$; $A = {}_{\mathcal{E}_n}[\text{Id}_{\mathbb{F}^n}]_{\mathcal{A}}$; $g : V \to \mathbb{F}^n$ is an isomorphism.
$\forall i \in \{1, \dots, n\}$, let $\mathbf{b}_i = g^{-1}(\mathbf{a}_i)$ and $\mathbf{c}_i = g^{-1}(\mathbf{e}_i)$. Then $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ are bases for V. Note also that ${}_{\mathcal{A}}[g]_{\mathcal{B}} = I_n$ and ${}_{\mathcal{C}}[g^{-1}]_{\mathcal{E}_n} = I_n$. We now have that:

$$\begin{aligned} {}_{\mathcal{C}}[\text{If}_V]_{\mathcal{B}} &= {}_{\mathcal{C}}[g^{-1} \circ \text{Id}_{\mathbb{F}^n} \circ g]_{\mathcal{B}} \\ &= {}_{\mathcal{C}}[g^{-1}]_{\mathcal{E}_n}\ {}_{\mathcal{E}_n}[\text{Id}_{\mathbb{F}^n}]_{\mathcal{A}}\ {}_{\mathcal{A}}[g]_{\mathcal{B}} \\ &= I_n A I_n \\ &= A. \end{aligned}$$

So, (i) holds. Q.E.D.

Proposition 4.2

Let U and V be non-trivial, finite-dimensional vector spaces over a field $\mathbb{F}$. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ be a basis of U, let $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be bases of V, and let $A \in \mathbb{F}^{n \times m}$ be a matrix. Then there exists a unique linear transformation $f : U \to V$ such that $A = {}_{\mathcal{C}}[f]_{\mathcal{B}}$.

Proof.

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$$[f_1(\mathbf{u})]_{\mathcal{C}} \;=\; {}_{\mathcal{C}}[f_1]_{\mathcal{B}}\,[\mathbf{u}]_{\mathcal{B}} \;=\; {}_{\mathcal{C}}[f_2]_{\mathcal{B}}\,[\mathbf{u}]_{\mathcal{B}} \;=\; [f_1(\mathbf{u})]_{\mathcal{C}},$$

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and consequently (since $[\cdot]_{\mathcal{C}} : V \to \mathbb{F}^n$ is an isomorphism), $f_1(\mathbf{u}) = f_2(\mathbf{u})$.

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Proof (continued). It remains to prove existence.

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$$\begin{aligned} {}_{\mathcal{C}}[f]_{\mathcal{B}} &= {}_{\mathcal{C}}[g_2 \circ h \circ g_1]_{\mathcal{B}} \\ &= {}_{\mathcal{C}}[g_2]_{\mathcal{E}_n}\ {}_{\mathcal{E}_n}[h]_{\mathcal{E}_m}\ {}_{\mathcal{E}_m}[g_1]_{\mathcal{B}} \\ &= I_n A I_m \\ &= A. \end{aligned}$$

This proves existence. Q.E.D.

Definition

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Theorem 4.3

Let n be a positive integer, let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $B, C \in \mathbb{F}^{n\times n}$ be matrices. Then the following are equivalent:

(i) there exists a linear transformation $f : V \to V$ and bases $\mathcal{B}$ and $\mathcal{C}$ of V such that $B = {}_{\mathcal{B}}[f]_{\mathcal{B}}$ and $C = {}_{\mathcal{C}}[f]_{\mathcal{C}}$;

(ii) B and C are similar.

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(ii) B and C are similar.

Proof. Suppose first that (i) holds, and fix a linear transformation $f : V \to V$ and bases $\mathcal{B}$ and $\mathcal{C}$ of V such that $B = {}_{\mathcal{B}}[f]_{\mathcal{B}}$ and $C = {}_{\mathcal{C}}[f]_{\mathcal{C}}$, as in the statement of (i).

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2. (ii) B and C are similar.

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$$\begin{aligned} B &= {}_{\mathcal{B}}[f]_{\mathcal{B}} \\ &= {}_{\mathcal{B}}[\text{Id}_V \circ f \circ \text{Id}_V]_{\mathcal{B}} \\ &= {}_{\mathcal{B}}[\text{Id}_V]_{\mathcal{C}} \ {}_{\mathcal{C}}[f]_{\mathcal{C}} \ {}_{\mathcal{C}}[\text{Id}_V]_{\mathcal{B}} \\ &= \Big({}_{\mathcal{B}}[\text{Id}_V]_{\mathcal{C}}\Big)^{-1} \underbrace{{}_{\mathcal{C}}[f]_{\mathcal{C}}}_{=C} \ {}_{\mathcal{C}}[\text{Id}_V]_{\mathcal{B}}. \end{aligned}$$

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Now for $P = {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$, we have $B = P^{-1}CP$. So, (ii) holds.

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Let n be a positive integer, let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $B, C \in \mathbb{F}^{n \times n}$ be matrices. Then the following are equivalent:

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2. (ii) B and C are similar.

Proof (continued). Suppose now that (ii) holds. Then fix an invertible matrix $P \in \mathbb{F}^{n \times n}$ such that $B = P^{-1}CP$. Since P is invertible, Proposition 4.1 guarantees that there exist bases $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ of V such that $P = {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$. Next, by Proposition 4.2, there exists a unique linear transformation $f : V \to V$ such that $C = {}_{\mathcal{C}}[f]_{\mathcal{C}}$.

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Proof (continued). Reminder: (ii) holds; $B = P^{-1}CP$; $P = {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$; $C = {}_{\mathcal{C}}[f]_{\mathcal{C}}$.

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$$\begin{aligned} B = P^{-1}CP &= \left({}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}\right)^{-1} {}_{\mathcal{C}}[f]_{\mathcal{C}}\ {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}} \\ &= {}_{\mathcal{B}}[\mathrm{Id}_V]_{\mathcal{C}}\ {}_{\mathcal{C}}[f]_{\mathcal{C}}\ {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}} \\ &= {}_{\mathcal{B}}[\mathrm{Id}_V \circ f \circ \mathrm{Id}_V]_{\mathcal{B}} \\ &= {}_{\mathcal{B}}[f]_{\mathcal{B}}. \end{aligned}$$

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We now have that $B = {}_{\mathcal{B}}[f]_{\mathcal{B}}$ and $C = {}_{\mathcal{C}}[f]_{\mathcal{C}}$. So, (i) holds.