Linear Algebra 2: Lecture 10

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1 A review of coordinate vectors and standard matrices

For a field $\mathbb{F}$, a positive integer n, and an index $i \in \{1, \dots, n\}$, the *i-th standard basis vector* in $\mathbb{F}^n$, denoted by $\mathbf{e}_i^n$, is the vector in $\mathbb{F}^n$ whose i-th entry is 1, and all of whose other entries are 0. $\mathcal{E}_n = \{\mathbf{e}_1^n, \dots, \mathbf{e}_n^n\}$ is called the *standard basis* of $\mathbb{F}^n$. When n is clear from context, we drop the superscript n and write $\mathbf{e}_i$ instead of $\mathbf{e}_i^n$.

Theorem 1.2 from Lecture Notes 4. *Let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a linear transformation. Then there exists a unique matrix A (called the* standard matrix of f*) such that for all $\mathbf{u} \in \mathbb{F}^m$, we have that $f(\mathbf{u}) = A\mathbf{u}$. Moreover, the standard matrix A of f is given by*

$$A \quad = \quad \left[\begin{array}{ccc} f(\mathbf{e}_1) & \dots & f(\mathbf{e}_m) \end{array} \right],$$

where $\mathbf{e}_1, \dots, \mathbf{e}_m$ are the standard basis vectors of $\mathbb{F}^m$.

For a positive integer n and a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of a vector space V over a field $\mathbb{F}$, the *coordinate vector* of a vector $\mathbf{v} \in V$ with respect to the basis $\mathcal{B}$ is the (unique) vector

$$[\mathbf{v}]_{\mathcal{B}} \quad := \quad \left[\begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_n \end{array} \right]$$

in $\mathbb{F}^n$ such that $\mathbf{v} = \alpha_1 \mathbf{b}_1 + \dots + \alpha_n \mathbf{b}_n$.

Note that if $\mathbb{F}$ is a field, then for all $\mathbf{u} \in \mathbb{F}^n$, we have that $[\mathbf{u}]_{\mathcal{E}_n} = \mathbf{u}$. Indeed, if $\mathbf{u} = \left[\begin{array}{ccc} u_1 & \dots & u_n \end{array} \right]^T$, then $\mathbf{u} = u_1\mathbf{e}_1 + \dots + u_n\mathbf{e}_n$, and consequently, $[\mathbf{u}]_{\mathcal{E}_n} = \left[\begin{array}{ccc} u_1 & \dots & u_n \end{array} \right]^T = \mathbf{u}$.

Theorem 2.2 from Lecture Notes 9. *Let U and V be vector spaces over a field $\mathbb{F}$, and assume that U is non-trivial (i.e. contains at least one non-zero vector) and finite-dimensional. Let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis of*

U, and let $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$.[1] *Then there exists a unique linear transformation $f : U \to V$ such that $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$. Moreover, this unique linear transformation $f : U \to V$ is defined as follows: for all $\mathbf{u} \in U$, we set*

$$f(\mathbf{u}) \quad = \quad \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n,$$

where $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ are such that $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 & \dots & \alpha_n \end{bmatrix}^T$.

2 Matrices of linear transformations

The following theorem generalizes Theorem 1.2 from Lecture Notes 4 (stated in section 1 above).

Theorem 2.1. *Let U and V be non-trivial, finite-dimensional vector spaces over a field $\mathbb{F}$. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ be a basis for U, let $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be a basis for V, and let $f : U \to V$ be a linear transformation. Then exists a unique matrix in $\mathbb{F}^{n \times m}$, denoted by ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ and called the* matrix of f with respect to $\mathcal{B}$ and $\mathcal{C}$, *such that for all $\mathbf{u} \in U$, we have that*

$${}_{\mathcal{C}}[f]_{\mathcal{B}} \, [\mathbf{u}]_{\mathcal{B}} \quad = \quad [f(\mathbf{u})]_{\mathcal{C}}.$$

Moreover, the matrix ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is given by

$${}_{\mathcal{C}}[f]_{\mathcal{B}} \quad = \quad \begin{bmatrix} [f(\mathbf{b}_1)]_{\mathcal{C}} & \dots & [f(\mathbf{b}_m)]_{\mathcal{C}} \end{bmatrix}.$$

Proof. We first prove existence. Fix $\mathbf{u} \in U$. We must show that

$$\begin{bmatrix} [f(\mathbf{b}_1)]_{\mathcal{C}} & \dots & [f(\mathbf{b}_m)]_{\mathcal{C}} \end{bmatrix} [\mathbf{u}]_{\mathcal{B}} \quad = \quad [f(\mathbf{u})]_{\mathcal{C}}$$

Set $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} \beta_1 & \dots & \beta_m \end{bmatrix}^T$, so that $\mathbf{u} = \beta_1 \mathbf{b}_1 + \dots + \beta_m \mathbf{b}_m$. We now have the following:

$$\begin{aligned}
\begin{bmatrix} [f(\mathbf{b}_1)]_{\mathcal{C}} & \dots & [f(\mathbf{b}_m)]_{\mathcal{C}} \end{bmatrix} [\mathbf{u}]_{\mathcal{B}} \quad &= \quad \begin{bmatrix} [f(\mathbf{b}_1)]_{\mathcal{C}} & \dots & [f(\mathbf{b}_m)]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} \\
&= \quad \beta_1 [f(\mathbf{b}_1)]_{\mathcal{C}} + \dots + \beta_m [f(\mathbf{b}_m)]_{\mathcal{C}} \\
&\overset{(*)}{=} \quad [\beta_1 f(\mathbf{b}_1) + \dots + \beta_m f(\mathbf{b}_m)]_{\mathcal{C}} \\
&\overset{(**)}{=} \quad [f(\beta_1 \mathbf{b}_1 + \dots + \beta_m \mathbf{b}_m)]_{\mathcal{C}} \\
&= \quad [f(\mathbf{u})]_{\mathcal{C}},
\end{aligned}$$

[1] Here, $\mathbf{v}_1, \dots, \mathbf{v}_n$ are arbitrary vectors in V. They are not necessarily pairwise distinct.

where (*) follows from the fact that $[\cdot]_{\mathcal{C}} : V \to \mathbb{F}^n$ is an isomorphism (and in particular, a linear transformation), and (**) follows from the fact that f is linear.

It remains to prove uniqueness. Fix any matrix $A \in \mathbb{F}^{n\times m}$ that has the property that for all $\mathbf{u} \in U$, we have that $A[\mathbf{u}]_{\mathcal{B}} = [f(\mathbf{u})]_{\mathcal{C}}$. We claim that $A = \begin{bmatrix} [f(\mathbf{b}_1)]_{\mathcal{C}} & \dots & [f(\mathbf{b}_m)]_{\mathcal{C}} \end{bmatrix}$. Set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$. We must show that $\mathbf{a}_1 = [f(\mathbf{b}_1)]_{\mathcal{C}}, \dots, \mathbf{a}_m = [f(\mathbf{b}_m)]_{\mathcal{C}}$. Fix $i \in \{1, \dots, m\}$. Note that $[\mathbf{b}_i]_{\mathcal{B}} = \mathbf{e}_i^m$, where as usual, $\mathbf{e}_i^m$ is the i-th standard basis vector in $\mathbb{F}^m$. But now

$$\mathbf{a}_i \quad = \quad A\mathbf{e}_i^m \quad = \quad A[\mathbf{b}_i]_{\mathcal{B}} \quad = \quad [f(\mathbf{b}_i)]_{\mathcal{C}}.$$

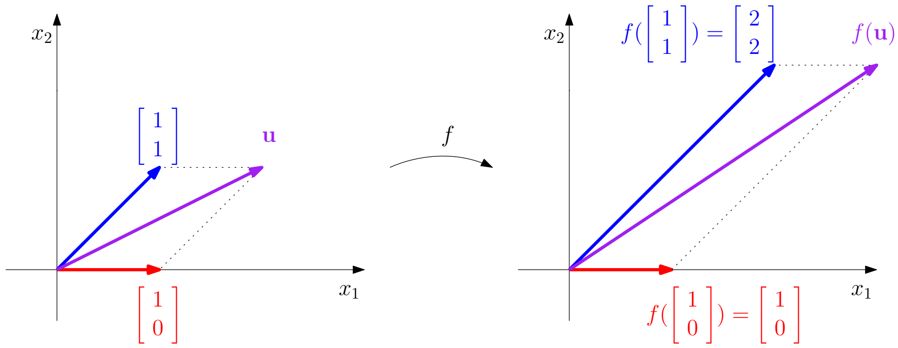
We have now shown that $A = \begin{bmatrix} [f(\mathbf{b}_1)]_{\mathcal{C}} & \dots & [f(\mathbf{b}_m)]_{\mathcal{C}} \end{bmatrix}$. This proves uniqueness, and we are done. □

Note that matrices of the form ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ are generalizations of standard matrices. Indeed, if $\mathbb{F}$ is a field and $f : \mathbb{F}^m \to \mathbb{F}^n$ is a linear transformation, then the matrix ${}_{\mathcal{E}_n}[f]_{\mathcal{E}_m}$ is precisely the standard matrix of f.

Example 2.2. *Consider the basis* $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ *for* $\mathbb{R}^2$.[2] *Consider the unique linear transformation* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *that has the property that*

- $f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$;
- $f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

Find the matrix ${}_{\mathcal{B}}[f]_{\mathcal{B}}$.



[2]Since RREF($\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$) $= I_2$, Theorem 4.1 from Lecture Notes 7 guarantees that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is indeed a basis for $\mathbb{R}^2$.

Solution. Using the formula from Theorem 2.1, we get that

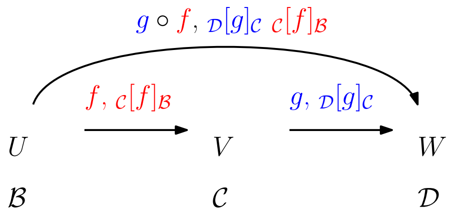
$$\begin{aligned} {}_{\mathcal{B}}[f]_{\mathcal{B}} &= \left[\begin{array}{cc} [f(\left[\begin{array}{c} 1 \\ 0 \end{array}\right])]_{\mathcal{B}} & [f(\left[\begin{array}{c} 1 \\ 1 \end{array}\right])]_{\mathcal{B}} \end{array}\right] \\ &= \left[\begin{array}{cc} [\left[\begin{array}{c} 1 \\ 0 \end{array}\right]]_{\mathcal{B}} & [\left[\begin{array}{c} 2 \\ 2 \end{array}\right]]_{\mathcal{B}} \end{array}\right] \\ &= \left[\begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array}\right]. \end{aligned}$$

□

What if we want to compute the standard matrix of the linear transformation f from Example 2.2? There is, indeed, a straightforward way to do this (see Example 2.8). However, we first need to develop some theory.

Proposition 2.3. *Let U, V, and W be non-trivial, finite-dimensional vector spaces over a field $\mathbb{F}$. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ be a basis for U, let $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be a basis for V, and let $\mathcal{D} = \{\mathbf{d}_1, \dots, \mathbf{d}_p\}$ be a basis for W. Let $f : U \to V$ and $g : V \to W$ be linear transformations. Then*

$$ {}_{\mathcal{D}}[g \circ f]_{\mathcal{B}} \;=\; {}_{\mathcal{D}}[g]_{\mathcal{C}} \; {}_{\mathcal{C}}[f]_{\mathcal{B}}. $$



Proof. By Theorem 2.1, ${}_{\mathcal{D}}[g \circ f]_{\mathcal{B}}$ is the unique matrix in $\mathbb{F}^{p \times m}$ such that for all $\mathbf{u} \in U$, we have that ${}_{\mathcal{D}}[g \circ f]_{\mathcal{B}} \, [\mathbf{u}]_{\mathcal{B}} = [(g \circ f)(\mathbf{u})]_{\mathcal{D}}$. So, it suffices to show that for all $\mathbf{u} \in U$, we have that $\Big({}_{\mathcal{D}}[g]_{\mathcal{C}} \; {}_{\mathcal{C}}[f]_{\mathcal{B}}\Big)[\mathbf{u}]_{\mathcal{B}} = [(g \circ f)(\mathbf{u})]_{\mathcal{D}}$. And indeed, for all $\mathbf{u} \in U$, we have that

$$\begin{aligned} \Big({}_{\mathcal{D}}[g]_{\mathcal{C}} \; {}_{\mathcal{C}}[f]_{\mathcal{B}}\Big)[\mathbf{u}]_{\mathcal{B}} &= {}_{\mathcal{D}}[g]_{\mathcal{C}}\Big({}_{\mathcal{C}}[f]_{\mathcal{B}} \; [\mathbf{u}]_{\mathcal{B}}\Big) \\ &= {}_{\mathcal{D}}[g]_{\mathcal{C}} \; [f(\mathbf{u})]_{\mathcal{C}} \\ &= [g(f(\mathbf{u}))]_{\mathcal{D}} \\ &= [(g \circ f)(\mathbf{u})]_{\mathcal{D}}. \end{aligned}$$

This completes the argument. □

We can use matrices of linear transformations to determine various properties of those linear transformations. For example, we have the following proposition.

Proposition 2.4. *Let U and V be non-trivial, finite-dimensional vector spaces over a field $\mathbb{F}$. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ be a basis for U, let $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be a basis for V, and let $f : U \to V$ be a linear transformation. Then all the following hold:*

(a) $\text{rank}(f) = \text{rank}\Big({}_{\mathcal{C}}[f]_{\mathcal{B}}\Big)$;[3]

(b) *f is an isomorphism if and only if ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is invertible (and in particular, square);*

(c) *if f is an isomorphism, then ${}_{\mathcal{B}}[f^{-1}]_{\mathcal{C}} = \Big({}_{\mathcal{C}}[f]_{\mathcal{B}}\Big)^{-1}$.*

Proof. We first prove (a). By Theorem 2.1, we have that

$$ {}_{\mathcal{C}}[f]_{\mathcal{B}} = \begin{bmatrix} [f(\mathbf{b}_1)]_{\mathcal{C}} & \dots & [f(\mathbf{b}_m)]_{\mathcal{C}} \end{bmatrix}. $$

We now compute:

$$ \begin{aligned} \text{rank}(f) &= \dim(\text{Im}(f)) \\ &\overset{(*)}{=} \dim\Bigg(\text{Span}\Big(f(\mathbf{b}_1), \dots, f(\mathbf{b}_m)\Big)\Bigg) \\ &\overset{(**)}{=} \dim\Bigg(\text{Span}\Big([f(\mathbf{b}_1)]_{\mathcal{C}}, \dots, [f(\mathbf{b}_m)]_{\mathcal{C}}\Big)\Bigg) \\ &= \dim\Bigg(\text{Col}\Big(\begin{bmatrix} [f(\mathbf{b}_1)]_{\mathcal{C}} & \dots & [f(\mathbf{b}_m)]_{\mathcal{C}} \end{bmatrix}\Big)\Bigg) \\ &= \dim\Bigg(\text{Col}\Big({}_{\mathcal{C}}[f]_{\mathcal{B}}\Big)\Bigg) \\ &= \text{rank}\Big({}_{\mathcal{C}}[f]_{\mathcal{B}}\Big), \end{aligned} $$

where (*) follows from the fact that $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ is a basis of U and from Proposition 2.1 from Lecture Notes 8, and (**) follows from the fact that $[\cdot]_{\mathcal{C}} : V \to \mathbb{F}^n$ is an isomorphism and from Proposition 3.7 from Lecture Notes 9. This proves (a).

[3] Recall that, by definition, $\text{rank}(f) = \dim\Big(\text{Im}(f)\Big)$.

We next prove (b). Suppose first that f is an isomorphism. Then $\dim(U) = \dim(V)$, i.e. $m = n$. In particular, ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is an $n \times n$ matrix. Next, since f is an isomorphism, we have that $\text{Im}(f) = V$, and so $\text{rank}(f) = \dim(\text{Im}(f)) = \dim(V) = n$. By (a), it follows that $\text{rank}({}_{\mathcal{C}}[f]_{\mathcal{B}}) = n$. Since ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is an $n \times n$ matrix of rank n, we know that ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is invertible.[4]

Suppose, conversely, that ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is invertible. In particular, ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is a square matrix, and it follows that $m = n$ (because ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is an $n \times m$ matrix). So, $\dim(U) = \dim(V)$. Moreover, since ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is an invertible $n \times n$ matrix, we have that $\text{rank}({}_{\mathcal{C}}[f]_{\mathcal{B}}) = n$. So, by (a), $\text{rank}(f) = n$, that is, $\dim(\text{Im}(f)) = n$. But now $\text{Im}(f)$ is an n-dimensional subspace of the n-dimensional vector space V, and it follows that $\text{Im}(f) = V$, i.e. f is onto V. But now $f : U \to V$ is an onto linear transformation between vector spaces of the same finite dimension; by Corollary 1.2 from Lecture Notes 9, it follows that f is also one-to-one. So, f is an isomorphism. This proves (b).

It remains to prove (c). Suppose that f is an isomorphism. Then by (b), ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is invertible. It now suffices to show that for all $\mathbf{v} \in V$, we have that $\left({}_{\mathcal{C}}[f]_{\mathcal{B}}\right)^{-1}[\mathbf{v}]_{\mathcal{C}} = [f^{-1}(\mathbf{v})]_{\mathcal{B}}$, for then Theorem 2.1 will imply that ${}_{\mathcal{B}}[f^{-1}]_{\mathcal{C}} = \left({}_{\mathcal{C}}[f]_{\mathcal{B}}\right)^{-1}$. So, fix $\mathbf{v} \in V$, and set $\mathbf{u} := f^{-1}(\mathbf{v})$. Then ${}_{\mathcal{C}}[f]_{\mathcal{B}}\ [\mathbf{u}]_{\mathcal{B}} = [f(\mathbf{u})]_{\mathcal{C}}$; since ${}_{\mathcal{C}}[f]_{\mathcal{B}}$ is invertible, we can multiply both sides by $\left({}_{\mathcal{C}}[f]_{\mathcal{B}}\right)^{-1}$ to obtain $[\mathbf{u}]_{\mathcal{B}} = \left({}_{\mathcal{C}}[f]_{\mathcal{B}}\right)^{-1}[f(\mathbf{u})]_{\mathcal{C}}$, i.e. $[f^{-1}(\mathbf{v})]_{\mathcal{B}} = \left({}_{\mathcal{C}}[f]_{\mathcal{B}}\right)^{-1}[\mathbf{v}]_{\mathcal{C}}$. This proves (c). □

Recall that for a set X, the identity function on X is denoted by Id_X. In other words, for a set X, the function $\text{Id}_X : X \to X$ is given by $\text{Id}_X(x) = x$ for all $x \in X$.

Given a non-trivial, finite-dimensional vector space V over a field $\mathbb{F}$, and bases $\mathcal{B}$ and $\mathcal{C}$ of V, we call the matrix ${}_{\mathcal{C}}[\text{Id}_V]_{\mathcal{B}}$ the *change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$* or the *transition matrix from $\mathcal{B}$ to $\mathcal{C}$*. Note that this matrix satisfies the property that for all $\mathbf{v} \in V$, we have that

$$ {}_{\mathcal{C}}[\text{Id}_V]_{\mathcal{B}}\ [\mathbf{v}]_{\mathcal{B}} \quad = \quad [\mathbf{v}]_{\mathcal{C}}. $$

Proposition 2.5. *Let V be a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be bases of V. Then the change of basis matrices ${}_{\mathcal{C}}[Id_V]_{\mathcal{B}}$ and ${}_{\mathcal{B}}[Id_V]_{\mathcal{C}}$ are invertible, and moreover, they are each other's inverses.*

Proof. Clearly, Id_V is an isomorphism, and moreover, $\text{Id}_V^{-1} = \text{Id}_V$. So, the result follows immediately from Proposition 2.4. □

[4] We are using Theorem 4.1 from Lecture Notes 7.

When our vector space V is of the form $\mathbb{F}^n$ (for some field $\mathbb{F}$), we get a nice formula for change of basis matrices (see Theorem 2.7 below). First, we need a lemma.

Lemma 2.6. *Let $\mathbb{F}$ be a field, let $\mathcal{E}_n = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{F}^n$, and let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be any basis for $\mathbb{F}^n$. Set $B := \begin{bmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_n \end{bmatrix}$. Then B is invertible, and moreover,*

$$ {}_{\mathcal{E}_n}[Id_{\mathbb{F}^n}]_{\mathcal{B}} \;=\; B \qquad \text{and} \qquad {}_{\mathcal{B}}[Id_{\mathbb{F}^n}]_{\mathcal{E}_n} \;=\; B^{-1}. $$

Proof. Let us first prove that ${}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{B}} = B$. In view of Theorem 2.1, it suffices to show that for all $\mathbf{v} \in \mathbb{F}^n$, we have that $B[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{E}_n}$. So, fix $\mathbf{v} \in \mathbb{F}^n$, and set $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} \beta_1 & \dots & \beta_n \end{bmatrix}^T$; then $\mathbf{v} = \beta_1\mathbf{b}_1 + \dots + \beta_n\mathbf{b}_n$. On the other hand, note that $[\mathbf{v}]_{\mathcal{E}_n} = \mathbf{v}$. Then

$$\begin{aligned} B[\mathbf{v}]_{\mathcal{B}} &= \begin{bmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \\ &= \beta_1\mathbf{b}_1 + \dots + \beta_n\mathbf{b}_n \\ &= \mathbf{v} \\ &= [\mathbf{v}]_{\mathcal{E}_n} \end{aligned}$$

This proves that ${}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{B}} = B$. The fact that B is invertible and that ${}_{\mathcal{B}}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{E}_n} = B^{-1}$ now follows from Proposition 2.5. □

Theorem 2.7. *Let $\mathbb{F}$ be a field, and let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be two bases for $\mathbb{F}^n$. Set $B := \begin{bmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_n \end{bmatrix}$ and $C := \begin{bmatrix} \mathbf{c}_1 & \dots & \mathbf{c}_n \end{bmatrix}$. Then*

$$ {}_{\mathcal{C}}[Id_{\mathbb{F}^n}]_{\mathcal{B}} \;=\; C^{-1}B. $$

Proof. We observe that

$$\begin{aligned} {}_{\mathcal{C}}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{B}} &= {}_{\mathcal{C}}[\mathrm{Id}_{\mathbb{F}^n} \circ \mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{B}} \\ &= {}_{\mathcal{C}}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{E}_n}\; {}_{\mathcal{E}_n}[\mathrm{Id}_{\mathbb{F}^n}]_{\mathcal{B}} && \text{by Proposition 2.3} \\ &= C^{-1}B && \text{by Lemma 2.6,} \end{aligned}$$

and we are done. □

Let us now return to the linear transformation f from Example 2.2. We would now like to find the standard matrix of this linear transformation.

Example 2.8. *Consider the basis* $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ *for* $\mathbb{R}^2$*. Consider the unique linear transformation* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *that has the property that*

- $f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$;
- $f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

Find the standard matrix of the linear transformation f*.*

Solution. In Example 2.2, we saw that ${}_{\mathcal{B}}[f]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Now, set $B := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and note that $B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Then the standard matrix of f is

$$\begin{aligned} {}_{\mathcal{E}_2}[f]_{\mathcal{E}_2} &= {}_{\mathcal{E}_2}[\text{Id}_{\mathbb{R}^2} \circ f \circ \text{Id}_{\mathbb{R}^2}]_{\mathcal{E}_2} \\ &= {}_{\mathcal{E}_2}[\text{Id}_{\mathbb{R}^2}]_{\mathcal{B}}\ {}_{\mathcal{B}}[f]_{\mathcal{B}}\ {}_{\mathcal{B}}[\text{Id}_{\mathbb{R}^2}]_{\mathcal{E}_2} && \text{by Proposition 2.3} \\ &= B\ {}_{\mathcal{B}}[f]_{\mathcal{B}}\ B^{-1} && \text{by Lemma 2.6} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}. \end{aligned}$$

Optional. Let us check that our answer is correct. Indeed, we have that

- $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = f(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$;
- $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = f(\begin{bmatrix} 1 \\ 1 \end{bmatrix})$.

So, our answer is correct. □

3 An example with polynomials

Example 3.1. *As usual, let* $\mathbb{P}^3_{\mathbb{Z}_2}$ *be the vector space (over* $\mathbb{Z}_2$*) of all polynomials with coefficients in* $\mathbb{Z}_2$ *and of degree at most* 3*.*

(a) Prove that there exists a unique linear transformation $f : \mathbb{P}^3_{\mathbb{Z}_2} \to \mathbb{P}_{\mathbb{Z}_2}$ *that satisfies all the following:*

- $f(1) = x^4$;
- $f(x+1) = x^3 + x^2$;
- $f(x^2+x+1) = x^2+1$;
- $f(x^3+x^2+x+1) = x$.

(b) Find rank(f), and determine whether f is one-to-one, where f is the linear transformation from part (a).

(c) Find the formula for the linear transformation $f : \mathbb{P}^3_{\mathbb{Z}_2} \to \mathbb{P}_{\mathbb{Z}_2}$ from part (a), that is, fill in the blank in the following:

$$f(a_3x^3 + a_2x^2 + a_1x + a_0) \quad = \quad \underline{\hspace{3cm}} \qquad \forall a_0, a_1, a_2, a_3 \in \mathbb{Z}_2.$$

Solution. In what follows, we consider the basis $\mathcal{A}_3 = \{1, x, x^2, x^3\}$ for $\mathbb{P}^3_{\mathbb{Z}_2}$, and the basis $\mathcal{A}_3 = \{1, x, x^2, x^3, x^4\}$ for $\mathbb{P}^4_{\mathbb{Z}_2}$

(a) It suffices to show that $\mathcal{B} = \{1, x+1, x^2+x+1, x^3+x^2+x+1\}$ is a basis for $\mathbb{P}^3_{\mathbb{Z}_2}$, for then Theorem 2.2 from Lecture Notes 9 will guarantee that there is a unique linear transformation $f : \mathbb{P}^3_{\mathbb{Z}_2} \to \mathbb{P}_{\mathbb{Z}_2}$ that satisfies the properties specified in the statement of (a).

We first form the coordinate vectors of the polynomials in $\mathcal{B}$, with respect to the basis $\mathcal{A}$ for $\mathbb{P}^3_{\mathbb{Z}_2}$:

- $[1]_{\mathcal{A}_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$;
- $[x+1]_{\mathcal{A}_3} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$;
- $[x^2+x+1]_{\mathcal{A}_3} = \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}^T$;
- $[x^3+x^2+x+1]_{\mathcal{A}_3} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$.

We now form the matrix

$$\begin{aligned} {}_{\mathcal{A}_3}[\mathrm{Id}_{\mathbb{P}^3_{\mathbb{Z}_2}}]_{\mathcal{B}} &= \begin{bmatrix} [1]_{\mathcal{A}_3} & [x+1]_{\mathcal{A}_3} & [x^2+x+1]_{\mathcal{A}_3} & [x^3+x^2+x+1]_{\mathcal{A}_3} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

By row reducing, we see that $\mathrm{RREF}(A) = I_4$. So, all four columns of A are pivot columns, and it follows that the columns of A form a basis for $\mathrm{Col}(A)$. Consequently, $\mathcal{B}$ forms a basis for $\mathrm{Span}(\mathcal{B})$. But now $\mathrm{Span}(\mathcal{B})$ is a 4-dimensional subspace of the 4-dimensional vector space $\mathbb{P}^3_{\mathbb{Z}_2}$, and it follows that $\mathrm{Span}(\mathcal{B}) = \mathbb{P}^3_{\mathbb{Z}_2}$. So, $\mathcal{B}$ is a basis of $\mathbb{P}^3_{\mathbb{Z}_2}$. As explained above, this implies that there is a unique linear transformation $f : \mathbb{P}^3_{\mathbb{Z}_2} \to \mathbb{P}_{\mathbb{Z}_2}$ that satisfies the properties specified in the statement of (a).

(b) Clearly, $\text{Im}(f)$ is a subspace of $\mathbb{P}^4_{\mathbb{Z}_2}$, and we have that

$$_{\mathcal{A}_4}[f]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

By row reducing, we see that

$$\text{RREF}\Big({}_{\mathcal{A}_4}[f]_{\mathcal{B}}\Big) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and so $\text{rank}({}_{\mathcal{A}_4}[f]_{\mathcal{A}_3}) = 4$. Therefore, by Proposition 2.4, $\text{rank}(f) = 4$. By the Rank-nullity theorem, we know that $\text{rank}(f) + \dim(\text{Ker}(f)) = \dim(\mathbb{P}^3_{\mathbb{Z}_2})$, and so $\dim(\text{Ker}(f)) = 0$. Therefore, $\text{Ker}(f) = \{0\}$, and it follows (by Theorem 1.1 from Lecture Notes 9) that f is one-to-one.

(c) Note that

$$\begin{aligned} {}_{\mathcal{A}_4}[f]_{\mathcal{A}_3} &= {}_{\mathcal{A}_4}[f \circ \text{Id}_{\mathbb{P}^3_{\mathbb{Z}_2}}]_{\mathcal{A}_3} \\ &= {}_{\mathcal{A}_4}[f]_{\mathcal{B}} \; {}_{\mathcal{B}}[\text{Id}_{\mathbb{P}^3_{\mathbb{Z}_2}}]_{\mathcal{A}_3} \\ &= {}_{\mathcal{A}_4}[f]_{\mathcal{B}} \; \Big({}_{\mathcal{A}_3}[\text{Id}_{\mathbb{P}^3_{\mathbb{Z}_2}}]_{\mathcal{B}}\Big)^{-1} \end{aligned}$$

We computed the matrix ${}_{\mathcal{A}_3}[\text{Id}_{\mathbb{P}^3_{\mathbb{Z}_2}}]_{\mathcal{B}}$ in part (a), and we computed the matrix ${}_{\mathcal{A}_4}[f]_{\mathcal{B}}$ in part (b). By row reducing the matrix $\left[\begin{array}{c|c} {}_{\mathcal{A}_3}[\text{Id}_{\mathbb{P}^3_{\mathbb{Z}_2}}]_{\mathcal{B}} & I_4 \end{array}\right]$, we obtain

$$\Big({}_{\mathcal{A}_3}[\text{Id}_{\mathbb{P}^3_{\mathbb{Z}_2}}]_{\mathcal{B}}\Big)^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We now compute

$$\begin{aligned}
{}_{\mathcal{A}_4}[f]_{\mathcal{A}_3} &= {}_{\mathcal{A}_4}[f]_{\mathcal{B}} \left({}_{\mathcal{A}_3}[\mathrm{Id}_{\mathbb{P}^3_{\mathbb{Z}_2}}]_{\mathcal{B}} \right)^{-1} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Now, for all $a_0, a_1, a_2, a_3 \in \mathbb{Z}_2$, we have that

$$\begin{aligned}
[f(a_3x^3 + a_2x^2 + a_1x + a_0)]_{\mathcal{A}_4} &= {}_{\mathcal{A}_4}[f]_{\mathcal{A}_3}\ [a_3x^3 + a_2x^2 + a_1x + a_0]_{\mathcal{A}_3} \\
&= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \\
&= \begin{bmatrix} a_2 + a_3 \\ a_3 \\ a_1 + a_3 \\ a_1 + a_2 \\ a_0 + a_1 \end{bmatrix}.
\end{aligned}$$

We now see that for all $a_0, a_1, a_2, a_3 \in \mathbb{Z}_2$, we have that

$$\begin{aligned}
f(a_3x^3 + a_2x^2 + a_1x + a_0) &= (a_0 + a_1)x^4 + (a_1 + a_2)x^3 + (a_1 + a_3)x^2 + \\
&\quad + a_3x + (a_2 + a_3).
\end{aligned}$$

Optional. We check that our formula for f is correct by checking that it indeed satisfies the conditions from part (a). Here, the equation (*) uses the formula that we obtained in part (c).

- $f(1) \overset{(*)}{=} (1+0)x^4 + (0+0)x^3 + (0+0)x^2 + 0x + (0+0) = x^4$;
- $f(x+1) \overset{(*)}{=} (1+1)x^4 + (1+0)x^3 + (1+0)x^2 + 0x + (0+0) = x^3 + x^2$;

- $f(x^2+x+1) \overset{(*)}{=} (1+1)x^4+(1+1)x^3+(1+0)x^2+0x+(1+0) = x^2+1$;
- $f(x^3+x^2+x+1) \overset{(*)}{=} (1+1)x^4+(1+1)x^3+(1+1)x^2+1x+(1+1) = x$.

As we can see, our formula is correct. □

4 A characterization of change of basis matrices. Similar matrices

As the following proposition shows, change of basis matrices are precisely the invertible matrices.

Proposition 4.1. *Let n be a positive integer, let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $A \in \mathbb{F}^{n\times n}$. Then the following are equivalent:*

(i) there exist bases $\mathcal{B}$ and $\mathcal{C}$ for V such that $A = {}_{\mathcal{C}}[Id_V]_{\mathcal{B}}$;

(ii) A is invertible.

Proof. Since $\text{Id}_{\mathbb{F}^n} : \mathbb{F}^n \to \mathbb{F}^n$ is an isomorphism, "(i) $\implies$ (ii)" follows from Proposition 2.4.

Suppose now that (ii) holds, i.e. that A is invertible. We must prove (i). Set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$. Since A is invertible, we know that $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is a basis of $\mathbb{F}^n$.[5] Then by Lemma 2.6, $A = {}_{\mathcal{E}_n}[\text{Id}_{\mathbb{F}^n}]_{\mathcal{A}}$. So far, we have shown that A is a change of basis matrix between two bases of $\mathbb{F}^n$. However, we need to show that A is a change of basis matrix between two bases of V. For this, we use an arbitrary isomorphism $g : V \to \mathbb{F}^n$ (the isomorphism g exists because V and $\mathbb{F}^n$ are both n-dimensional vector spaces over $\mathbb{F}$). For all $i \in \{1, \dots, n\}$, let $\mathbf{b}_i = g^{-1}(\mathbf{a}_i)$ and $\mathbf{c}_i = g^{-1}(\mathbf{e}_i)$. Since g is an isomorphism, so is g^{-1}. Since $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ and $\mathcal{E}_n = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ are bases for $\mathbb{F}^n$, Proposition 3.7 from Lecture Notes 9 now guarantees that $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ are bases for V. Note also that ${}_{\mathcal{A}}[g]_{\mathcal{B}} = I_n$ and ${}_{\mathcal{C}}[g^{-1}]_{\mathcal{E}_n} = I_n$.[6] We now have that

$$\begin{aligned} {}_{\mathcal{C}}[\text{If}_V]_{\mathcal{B}} &= {}_{\mathcal{C}}[g^{-1} \circ \text{Id}_{\mathbb{F}^n} \circ g]_{\mathcal{B}} \\ &= {}_{\mathcal{C}}[g^{-1}]_{\mathcal{E}_n} \; {}_{\mathcal{E}_n}[\text{Id}_{\mathbb{F}^n}]_{\mathcal{A}} \; {}_{\mathcal{A}}[g]_{\mathcal{B}} \\ &= I_n A I_n \\ &= A. \end{aligned}$$

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[5] This follows from Theorem 4.1 from Lecture Notes 7.

[6] Indeed,

- ${}_{\mathcal{A}}[g]_{\mathcal{B}} = \begin{bmatrix} [g(\mathbf{b}_1)]_{\mathcal{A}} & \dots & [g(\mathbf{b}_n)]_{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} [\mathbf{a}_1]_{\mathcal{A}} & \dots & [\mathbf{a}_n]_{\mathcal{A}} \end{bmatrix} = I_n$;
- ${}_{\mathcal{C}}[g^{-1}]_{\mathcal{E}_n} = \begin{bmatrix} [g^{-1}(\mathbf{e}_1)]_{\mathcal{C}} & \dots & [g^{-1}(\mathbf{e}_n)]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} [\mathbf{c}_1]_{\mathcal{C}} & \dots & [\mathbf{c}_n]_{\mathcal{C}} \end{bmatrix} = I_n$.

So, (i) holds. $\square$

The following proposition is also useful. It is, in a sense, the converse of Theorem 2.1.

Proposition 4.2. *Let U and V be non-trivial, finite-dimensional vector spaces over a field $\mathbb{F}$. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ be a basis of U, let $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be bases of V, and let $A \in \mathbb{F}^{n\times m}$ be a matrix. Then there exists a unique linear transformation $f : U \to V$ such that $A = {}_{\mathcal{C}}[f]_{\mathcal{B}}$.*

Proof. We first prove uniqueness. Suppose that $f_1, f_2 : U \to V$ are linear transformations such that $A = {}_{\mathcal{C}}[f_1]_{\mathcal{B}}$ and $A = {}_{\mathcal{C}}[f_1]_{\mathcal{B}}$. We must show that $f_1 = f_2$. Note that for all $\mathbf{u} \in U$, we have that

$$[f_1(\mathbf{u})]_{\mathcal{C}} \quad = \quad {}_{\mathcal{C}}[f_1]_{\mathcal{B}}\,[\mathbf{u}]_{\mathcal{B}} \quad = \quad {}_{\mathcal{C}}[f_2]_{\mathcal{B}}\,[\mathbf{u}]_{\mathcal{B}} \quad = \quad [f_1(\mathbf{u})]_{\mathcal{C}},$$

and consequently (since $[\cdot]_{\mathcal{C}} : V \to \mathbb{F}^n$ is an isomorphism), $f_1(\mathbf{u}) = f_2(\mathbf{u})$. So, $f_1 = f_2$. This proves uniqueness.

It remains to prove existence. Let $h : \mathbb{F}^m \to \mathbb{F}^n$ be given by $h(\mathbf{x}) = A\mathbf{x}$. Then h is linear and its standard matrix is A, i.e. ${}_{\mathcal{E}_n}[h]_{\mathcal{E}_m} = A$. Now, let $g_1 : U \to \mathbb{F}^m$ be the unique linear transformation such that $g_1(\mathbf{b}_1) = \mathbf{e}_1^m, \dots, g_1(\mathbf{b}_1) = \mathbf{e}_m^m$,[7] and let $g_2 : \mathbb{F}^n \to V$ be the unique linear transformation such that $g_2(\mathbf{e}_1^n) = \mathbf{c}_1, \dots, g_2(\mathbf{e}_n^n) = \mathbf{c}_n$.[8] Note that ${}_{\mathcal{E}_m}[g_1]_{\mathcal{B}} = I_m$ and ${}_{\mathcal{C}}[g_2]_{\mathcal{E}_n} = I_n$. Set $f := g_2 \circ h \circ g_1$. Then $f : U \to V$ is a linear transformation, and we have that

$$\begin{aligned} {}_{\mathcal{C}}[f]_{\mathcal{B}} &= {}_{\mathcal{C}}[g_2 \circ h \circ g_1]_{\mathcal{B}} \\ &= {}_{\mathcal{C}}[g_2]_{\mathcal{E}_n}\ {}_{\mathcal{E}_n}[h]_{\mathcal{E}_m}\ {}_{\mathcal{E}_m}[g_1]_{\mathcal{B}} \\ &= I_n A I_m \\ &= A. \end{aligned}$$

This proves existence, and we are done. $\square$

Given a field $\mathbb{F}$, we say that matrices $A, B \in \mathbb{F}^{n\times n}$ are *similar* if there exists an invertible matrix $P \in \mathbb{F}^{n\times n}$ such that $A = P^{-1}BP$.[9]

As the following theorem shows, two matrices are similar if and only if they represent the same linear transformation, but possibly with respect to different bases.

[7] Since $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ is a basis of U, the existence and uniqueness of g_1 follows from Theorem 2.2 from Lecture Notes 9, stated in section 1 (above).

[8] Since $\mathcal{E}_n = \{\mathbf{e}_1^n, \dots, \mathbf{e}_n^n\}$ is a basis of $\mathbb{F}^n$, the existence and uniqueness of g_2 follows from Theorem 2.2 from Lecture Notes 9, stated in section 1 (above).

[9] Note that if $A = P^{-1}BP$, then $B = PAP^{-1} = (P^{-1})^{-1}AP^{-1}$. So, matrix similarity is symmetric, as we would expect. It is also easy to see that it is reflexive and transitive (proof?).

Theorem 4.3. *Let n be a positive integer, let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $B, C \in \mathbb{F}^{n \times n}$ be matrices. Then the following are equivalent:*

(i) there exists a linear transformation $f : V \to V$ and bases $\mathcal{B}$ and $\mathcal{C}$ of V such that $B = {}_{\mathcal{B}}[f]_{\mathcal{B}}$ and $C = {}_{\mathcal{C}}[f]_{\mathcal{C}}$;

(ii) B and C are similar.

Proof. Suppose first that (i) holds, and fix a linear transformation $f : V \to V$ and bases $\mathcal{B}$ and $\mathcal{C}$ of V such that $B = {}_{\mathcal{B}}[f]_{\mathcal{B}}$ and $C = {}_{\mathcal{C}}[f]_{\mathcal{C}}$, as in the statement of (i). Then

$$\begin{aligned} B &= {}_{\mathcal{B}}[f]_{\mathcal{B}} \\ &= {}_{\mathcal{B}}[\mathrm{Id}_V \circ f \circ \mathrm{Id}_V]_{\mathcal{B}} \\ &= {}_{\mathcal{B}}[\mathrm{Id}_V]_{\mathcal{C}}\ {}_{\mathcal{C}}[f]_{\mathcal{C}}\ {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}} \\ &= \Big({}_{\mathcal{B}}[\mathrm{Id}_V]_{\mathcal{C}}\Big)^{-1}\ \underbrace{{}_{\mathcal{C}}[f]_{\mathcal{C}}}_{=C}\ {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}. \end{aligned}$$

We now set $P = {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$, and we observe that $B = P^{-1}CP$. So, B and C are similar, that is, (ii) holds.

Suppose now that (ii) holds. Then fix an invertible matrix $P \in \mathbb{F}^{n \times n}$ such that $B = P^{-1}CP$. Since P is invertible, Proposition 4.1 guarantees that there exist bases $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ of V such that $P = {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}$. Next, by Proposition 4.2, there exists a unique linear transformation $f : V \to V$ such that $C = {}_{\mathcal{C}}[f]_{\mathcal{C}}$. But now

$$\begin{aligned} B = P^{-1}CP &= \Big({}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}}\Big)^{-1}\ {}_{\mathcal{C}}[f]_{\mathcal{C}}\ {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}} \\ &= {}_{\mathcal{B}}[\mathrm{Id}_V]_{\mathcal{C}}\ {}_{\mathcal{C}}[f]_{\mathcal{C}}\ {}_{\mathcal{C}}[\mathrm{Id}_V]_{\mathcal{B}} \\ &= {}_{\mathcal{B}}[\mathrm{Id}_V \circ f \circ \mathrm{Id}_V]_{\mathcal{B}} \\ &= {}_{\mathcal{B}}[f]_{\mathcal{B}} \end{aligned}$$

We now have that $B = {}_{\mathcal{B}}[f]_{\mathcal{B}}$ and $C = {}_{\mathcal{C}}[f]_{\mathcal{C}}$. So, (i) holds. □