Linear Algebra 2:
HW#8

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due Friday, May 19, 2022, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Alternatively (if you don't feel like typing or scanning), you may submit a hard copy of your HW in lecture or tutorial **before** the deadline. Please do **not** e-mail your HW to me. Please write your **name** on the top of the first page of your HW.

Problem 1 (30 points). *Let V be a vector space over a field $\mathbb{F}$, let $q : V \to \mathbb{F}$ be a quadratic form on V, and let $\mathbf{v} \in V$ be a fixed vector. Consider the function $L : V \to \mathbb{F}$ given by*

$$L(\mathbf{x}) \quad = \quad q(\mathbf{x}+\mathbf{v}) - q(\mathbf{x}) - q(\mathbf{v})$$

for all $\mathbf{x} \in V$. Prove that L is linear.

Problem 2 (30 points). *Consider the quadratic form $q : \mathbb{R}^2 \to \mathbb{R}$ given by*

$$q(\mathbf{x}) \quad = \quad 6x_1^2 + 4x_1x_2 + 3x_2^2$$

*for all $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ in $\mathbb{R}^2$. Find a **polar** basis $\mathcal{B}$ of $\mathbb{R}^2$ for the quadratic form q, and find the matrix B of q with respect to $\mathcal{B}$. (So, the matrix B should be diagonal, and all its entries on the main diagonal should be 1, -1, or 0.)*

> ***Hint:*** *Read the proof of Sylvester's law of inertia (from Lecture Notes 22), and compute accordingly.*

Problem 3 (40 points)**.** *Let* V *be a vector space over a field* $\mathbb{F}$*, and let* $M_1 = \mathbf{a}_1 + U_1$ *and* $M_2 = \mathbf{a}_2 + U_2$ *be affine subspaces of* V *(where* $\mathbf{a}_1, \mathbf{a}_2$ *are vectors and* U_1, U_2 *are linear subspaces of* V*). Assume that* $U_1 \subseteq U_2$.

(a) Prove that if $\mathbf{a}_1 - \mathbf{a}_2 \in U_2$*, then* $M_1 \subseteq M_2$.

(b) Prove that if $\mathbf{a}_1 - \mathbf{a}_2 \notin U_2$*, then* $M_1 \cap M_2 = \emptyset$.