Linear Algebra 2: HW#5

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due Friday, April 14, 2022, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Alternatively (if you don't feel like typing or scanning), you may submit a hard copy of your HW in lecture or tutorial **before** the deadline. Please do **not** e-mail your HW to me. Please write your **name** on the top of the first page of your HW.

Problem 1 (20 points)**.** *Let a, b, c be positive real numbers. Using determinants, compute the volume of the solid enclosed by the ellipsoid*

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1 \right\}.$$

Problem 2 (20 points)**.** *For what values of complex constants a, b, c does $\mathbb{C}^3$ have a basis formed by eigenvectors of the matrix*

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 2 \end{bmatrix}?$$

Make sure you prove that your answer is correct.

Problem 3 (20 points)**.** *Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Prove that A is invertible if and only if 0 is* ***not*** *an eigenvalue of A.*

Problem 4 (20 points). *Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$ be an invertible matrix. Let $\mathbf{x}$ be an eigenvector of A associated with an eigenvalue λ of A (by Problem 3, we have that $\lambda \neq 0$). Prove that $\mathbf{x}$ is an eigenvector of A^{-1}. What is the eigenvalue of A^{-1} that the eigenvector $\mathbf{x}$ is associated with?*

Problem 5 (20 points). *Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix, i.e. one that satisfies $A^T = A$.*

(a) Prove that for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$, we have that $(A\mathbf{v}_1) \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot (A\mathbf{v}_2)$.

(b) Prove that if $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$ are eigenvectors of A associated with distinct eigenvalues of A (say, λ_1 and λ_2, respectively), then $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.