Linear Algebra 2:
HW#4

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due Friday, ~~March 31~~ April 7, 2022, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Alternatively (if you don't feel like typing or scanning), you may submit a hard copy of your HW in lecture or tutorial **before** the deadline. Please do **not** e-mail your HW to me. Please write your **name** on the top of the first page of your HW.

Problem 1 (25 points). *Consider the vectors*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}$$

in $\mathbb{R}^4$, *and set* $C := Span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$.

(a) *[15 points] Find an orthonormal basis for* C *and an orthonormal basis for* $C^\perp$.

(b) *[10 points] Let* $\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$. *Compute vectors* $\mathbf{y} \in C$ *and* $\mathbf{z} \in C^\perp$ *such that* $\mathbf{x} = \mathbf{y} + \mathbf{z}$.

Problem 2 (25 points). *Let* $\mathbb{F}$ *be a field, let* $n \geq 2$ *be an integer, and let* $\mathbf{a}_2, \dots, \mathbf{a}_n$ *be linearly independent vectors in* $\mathbb{F}^n$. *Define the function* $f : \mathbb{F}^n \to \mathbb{F}$ *by setting*

$$f(\mathbf{x}) = \begin{vmatrix} \mathbf{x} & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{vmatrix} \qquad \text{for all } \mathbf{x} \in \mathbb{F}^n.$$

By Proposition 3.1(a) from Lecture Notes 15, f *is linear. Find a basis for* $Ker(f)$, *and compute the dimension of* $Ker(f)$ *and* $Im(f)$. *Make sure you fully justify your answer.*

Problem 3 (25 points)**.** *Let A be an invertible matrix in $\mathbb{R}^n$, and assume that the entries of A are all integers. Prove that the entries of A^{-1} are all integers if and only if $det(A) = \pm 1$.*

> ***Hint/Remark:*** *You will need to use Theorem 4.2 from Lecture Notes 16, among other things.* ~~*(We have not yet gone over the proof of Theorem 4.2 in class, but we did see the theorem statement.)*~~ *[We have now seen the proof of Theorem 4.2.]*

Problem 4 (25 points)**.** *Let n be a positive integer, let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $f : V \to V$ be a linear transformation. Define the determinant of f by setting*

$$det(f) \quad = \quad det\Big({}_{\mathcal{B}}[f]_{\mathcal{B}}\Big),$$

where $\mathcal{B}$ is any basis of V. Prove that $det(f)$ is well defined, i.e. that it does not depend on the particular choice of basis $\mathcal{B}$.

> ***Hint/Remark:*** *So, you need to show that if $\mathcal{B}_1$ and $\mathcal{B}_2$ are any two bases of V, then $det\Big({}_{\mathcal{B}_1}[f]_{\mathcal{B}_1}\Big) = det\Big({}_{\mathcal{B}_2}[f]_{\mathcal{B}_2}\Big)$.*